Finslerian angle-preserving connection in two-dimensional case. Regular realization

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Abstract

We show that the metrical connection can be introduced in the two-dimensional Finsler space such that entailed parallel transports along curves joining points of the underlying manifold keep the two-vector angle as well as the length of the tangent vector, thereby realizing isometries of tangent spaces under the parallel transports. The curvature tensor is found. In case of the Finsleroid-regular space, constructions possess the C^∞-regular status globally regarding the dependence on tangent vectors. Many involved and important relations are explicitly derived.

Key words: Finsler metrics, angle, connection, curvature tensors.

1. Motivation and description

During all the history of development of the Finsler geometry the notion of connection was attracted sincere and great attention of investigators devoted to general theory as well as to specialized applications. The methods of construction of connection are founded upon setting forth a convenient system of axioms. Various standpoints were taken to get deeper insights into the notion (see [1-5]).

The general idea underlining the present work is to set forth the requirement that the connection be compatible with the preservation of the two-vector angle under the parallel transports of vectors.

The notion of angle is of key significance in geometry. In the field of two-dimensional Finsler spaces the angle between two vectors of a given tangent space can naturally be measured by the area of the domain bounded by the vectors and the indicatrix arc. The theorem can be proved which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angles thus appeared. This fundamental Tamássy's theorem [6], which explains us that the angle structure fixes the metric structure in the Finsler space, gives rise to the following important question: *Does the angle structure also generate the connection?* The present work proposes a positive and explicit answer, confining the Finslerian consideration to the two-dimensional case.

Let M be a C^∞-differentiable 2-dimensional manifold, T_xM denote the tangent space supported by the point $x \in M$, and $y \in T_xM\backslash 0$ mean tangent vectors. Given a Finsler metric function $F = F(x,y)$, we obtain the two-dimensional Finsler space $\mathcal{F}_2 = (M,F)$.

We shall use the standard Finslerian notation for local components $l^k = y^k/F$, $y_k = F\partial F/\partial y^k \equiv g_{kn}y^n$, $g_{ij} = \partial y_i/\partial y^j$ of the unit vector, the covariant tangent vector, and the Finsler metric tensor, respectively. The covariant components $l_k = g_{kn}l^n$ can be obtained from $l_k = \partial F/\partial y^k$. By means of the contravariant components g^{ij} the Cartan tensor $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$ can be contracted to yield the vector $C_k = g^{ij}C_{ijk}$. It is convenient to use the tensor $A_{ijk} = FC_{ijk}$ and the vector $A_k = FC_k = g^{ij}A_{ijk}$. The indices $i, j, ...$ are specified over the range (1,2). The square root $\sqrt{}$ stands always in the positive sense. It is often convenient to apply the expansion $A_{ijk} = Im_im_jm_k$ in terms of the m_i obtainable from $g_{ij} = l_il_j + m_im_j$, where I thus appeared is the so-called main scalar. Our consideration will be of local nature, unless otherwise is stated explicitly.

To each point $x \in M$, the Finsler space $\mathcal{F}_2$ associates the tangent Riemannian space, to be denoted by $\mathcal{R}_{\{x\}} := \{T_xM, g_{ij}(x,y)\}$, in which x is treated fixed and $y \in T_xM$ is variable. In the Riemannian space the $\mathcal{R}_{\{x\}}$ reduces to the tangent Euclidean space. The remarkable and well-known property of the Riemannian Levi Civita connection is that the entailed parallel transports along curves drawn on the underlined manifold keep the length of the tangent vectors and produce the isometric mapping of the tangent Euclidean spaces.

We show that these two fundamental Riemannian properties can successfully be extended to operate in the Finsler space $\mathcal{F}_2$. Namely, if sufficient smoothness holds then it proves possible to introduce the respective connection coefficients $\{N^k_i(x,y), D^k{}_{in}(x,y)\}$ in a simple and explicit way. The coefficients $N^k_i(x,y)$ are required to construct the conventional operator $d_n = \partial_{x^n} + N^k_n(x,y)\partial_{y^k}$, where $\partial_{x^n} = \partial/\partial x^n$ and $\partial_{y^m} = \partial/\partial y^m$. The keeping of the Finsler length of the tangent vectors means $d_nF = 0$. Let us attract also the angle function $\theta = \theta(x,y)$ (to measure the length ds of infinitesimal arc on the indicatrix by $d\theta$) and raise forth the requirement that $d_n\theta = k_n$ with a covariant vector field $k_n = k_n(x)$. If this is fulfilled, then for pairs $\{\theta_1 = \theta(x,y_1), \theta_2 = \theta(x,y_2)\}$ we obtain

the nullification $d_n(\theta_2-\theta_1)=0$ which tells us that the *preservation of the two-vector angle* $\theta_2-\theta_1$ holds true under the parallel transports initiated by the coefficients $N^k_i(x,y)$.

There arise the coefficients $D^k{}_{in}(x,y)=-N^k{}_{in}(x,y)$ with $N^k{}_{in}(x,y)=\partial N^k_i(x,y)/\partial y^n$. A careful analysis has shown that a simple and attractive proposal of the coefficients $N^k_i(x,y)$ (namely, (2.6) of Section 2) can be made such that nullifications $d_iF=0$ and $d_i\theta-k_i=0$ are simultaneously satisfied, and also the vanishing $l_kN^k{}_{nmi}=0$ holds fine because of the representation $FN^k{}_{nmi}=-A^k{}_{mi}d_n\ln|I|$ entailed (see (2.14)), where $N^k{}_{nmi}=\partial N^k{}_{nm}/\partial y^i$, so that the action of the arisen covariant derivative on the involved Finsler metric tensor yields just the zero. The coefficients $D^k{}_{in}$ are not symmetric in the subscripts i,n.

Having realized this program (in Section 2), we feel sure that the arisen mappings of the space $\mathcal{R}_{\{x\}}$ under the respective parallel transports along the curves running on M are isometries.

The coefficients $\{N^k_i(x,y),D^k{}_{in}(x,y)\}$ obtained in this way are *not* constructed from the Finsler metric tensor and derivatives of the tensor. This circumstance may be estimated to be a cardinal distinction of the Finsler connection induced by the angle structure from the conventional Riemannian precursor which exploits the Riemannian Christoffel symbols to be the coefficients $D^k{}_{in}$. The structure of the coefficients N^k_n involves the derivative $\partial\theta/\partial x^n$ on the equal footing with the derivative $\partial F/\partial x^n$ (see (2.6) in Section 2).

The involved vector field $k_i=k_i(x)$ may be taken arbitrary. However, the field can be specified if the Riemannian limit of the connection proposed is attentively considered. Indeed, in the Riemannian limit the connection coefficients $D^k{}_{nh}(x,y)$ reduce to the coefficients $\bar L^k{}_{nh}=-L^k{}_{nh}=\bar L^k{}_{nh}(x)$ which are *not symmetric* with respect to the subscripts (see (4.6)). If we want to obtain the torsionless coefficients, like to the Riemannian geometry proper, we must make the choice $k_n=-n^h\nabla_n\widetilde{b}_h$ in accordance with (4.16), where ∇_n is the Riemannian covariant derivative taken with the Christoffel symbols $a^k{}_{nh}$. The $\widetilde{b}_h=\widetilde{b}_h(x)$ is a vector field chosen to fulfill $\theta(x,\widetilde{b}(x))=0$, and the pair $n_i,\widetilde{b}_i$ is orthonormal. With the choice we obtain $\bar L^k{}_{nh}=a^k{}_{nh}$, thereby completely specifying the coefficients $\{N^k_i,D^k{}_{in}\}$.

It is appropriate to construct the *osculating* Riemannian metric tensor along the vector field $\widetilde{b}^i=\widetilde{b}^i(x)$ and introduce the *θ-associated Riemannian space* to compare the Finsler properties of the space $\mathcal{F}_2$ with the properties of the Riemannian precursor.

In Section 2 the required coefficients N^k_n are proposed and nearest implications are indicated. By the help of the identities $\partial m^k/\partial y^m=-(Im^k+l^k)m_m/F$ and $\partial m_m/\partial y^i=(Im_m-l_m)m_i/F$, the validity of the vanishing $l_kN^k{}_{nmi}=0$ can readily be verified. The angle function θ introduced does measure the length of the indicatrix arc according to $ds=d\theta$ (see (2.23)). We derive also the equality $\Sigma_{\{y_1,y_2\}}=(1/2)(\theta_2-\theta_1)$, where the left-hand side is the area of the sector bounded by the vectors y_1,y_2 and the indicatrix arc (see (2.27)). The equality demonstrates clearly that, in context of the two-dimensional theory to which our treatment is restricted, the method of introduction of angle by the help of the function θ is equivalent to the method founded in [6] on the notion of area. We are entitled therefore to raise the thesis that, in such a context, the angle-preserving connection is tantamount to the area-preserving connection.

In Section 3 we show how the curvature tensor $\rho_k{}^n{}_{ij}$ of the space $\mathcal{F}_2$ can be explicated from the commutator of the covariant derivative arisen, yielding the astonishingly simple representation (3.14).

In Section 4 we outline Riemannian counterparts. It appears that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij}$ is factorable, in accordance with (4.19).

In Conclusions several ideas are emphasized.

In Appendix A the validity of succession of important relations appeared in the analysis has been explicitly demonstrated.

The possibility of global realization of the angle-preserving connection implies high regularity properties of the Finsler metric function and the angle function. Such a lucky possibility occurs in the Finsleroid-regular space $\mathcal{FR}^{PD}_{g;c}$ (introduced and studied in [7-10]). Appendix B is devoted to the space. The Finsler metric function $K = K(x, y)$ of the space $\mathcal{FR}^{PD}_{g;c}$ involves a Riemannian metric tensor a_{mn} and the vector field $b^n = b^n(x)$ which represents the distribution of axis of indicatrix. We have two scalars, namely the characteristic scalar $g = g(x)$ and the norm $c = c(x) = ||b|| \equiv \sqrt{a^{mn}b_m b_n}$ of the 1-form $b = b_i(x)y^i$. The metric function K is not absolute homogeneous.

Simple direct calculation shows that the partial derivative $\partial K/\partial x^n$ obeys the total regularity with respect to the vector variable y (see (B.59)). The same regularity is shown by the partial derivative $\partial\theta/\partial x^n$ of the involved angle function $\theta = \theta(x, y)$ (see (B.83)). Therefore, all the ingredients in the coefficients N^k_n of the form proposed by (2.6) are of this high regularity. Thus we observe the remarkable phenomenon that the space $\mathcal{FR}^{PD}_{g;c}$ possesses the angle-preserving connection of the C^∞-regular status globally regarding the y-dependence. All the formulas appeared in Appendix B are such that their right-hand parts are of this high regularity status. Arbitrary (smooth) dependence on x in $g = g(x)$, $b_i = b_i(x)$, and $a_{ij} = a_{ij}(x)$ is permitted. The particular condition $c = const$ simplifies many representations. With $c = const$ we make an attractive proposal (B.85) for the coefficients N^k_n and demonstrate in the step-by-step way that all the three conditions listed in (2.8) are obtained and the representation of the announced form (2.6) is fulfilled. The curvature tensor has been evaluated, yielding the representations of the forms (3.15) and (4.19) with the global C^∞-regular status of dependence on y. Using an appropriate regular atlas of charts in the space $\mathcal{R}_{\{x\}} := \{T_xM, g_{ij}(x, y)\}$, it proves possible to verify that over all the space $T_xM\backslash 0$ the function $\theta = \theta(x, y)$ is smooth of class C^∞ with respect to y. The entailed two-vector angle $\theta_2 - \theta_1$ is symmetric and additive. The θ is represented by means of integral and is not obtainable through composition of elementary functions. Several representations essential for evaluations are indicated.

Quite similar evaluation can be performed for the Randers metric function (Appendix C), yielding again the angle-preserving connection of the C^∞-regular status globally regarding the y-dependence.

2. Proposal of connection coefficients

It is convenient to proceed with the skew-symmetric tensorial object $\epsilon_{ik} = \sqrt{\det(g_{mn})}\,\gamma_{ik}$, where $\gamma_{11} = \gamma_{22} = 0$ and $\gamma_{12} = -\gamma_{21} = 1$, to construct

$$m_i = -\epsilon_{ik} l^k. \tag{2.1}$$

The angular metric tensor $h_{ij} = g_{ij} - l_i l_j$ and the Cartan tensor C_{ijk} are factorized, and the Finsler metric tensor g_{ij} is expanded, according to

$$h_{ij} = m_i m_j, \qquad A_{ijk} = I m_i m_j m_k, \qquad g_{ij} = l_i l_j + m_i m_j. \tag{2.2}$$

It is also convenient to introduce the $\theta = \theta(x,y)$ by the help of the equation

$$F\frac{\partial\theta}{\partial y^n} = m_n, \tag{2.3}$$

assuming that the function θ is positively homogeneous of degree zero with respect to the variable y. These formulas are known from Section 6.6 of the book [1]. We denote the main scalar by I, instead of J used in the book. Our θ is the φ of Section 6.6 of [1]. The object ϵ_{ik} is a pseudo-tensor, whence m_i is a pseudo-vector and I,θ are pseudo-scalars. However, we don't consider the coordinate reflections and, therefore, we are entitled to refer to these objects as to "the vector m_i" and "the scalars I,θ".

We need the coefficients $N^k_n = N^k_n(x,y)$ to construct the operator

$$d_n = \frac{\partial}{\partial x^n} + N^k_n\frac{\partial}{\partial y^k} \tag{2.4}$$

which generates a covariant vector d_nW when is applied to an arbitrary differentiable scalar $W = W(x,y)$. We shall use also the derivative coefficients

$$N^k{}_{nm} = \frac{\partial N^k_n}{\partial y^m}, \qquad N^k{}_{nmi} = \frac{\partial N^k{}_{nm}}{\partial y^i}. \tag{2.5}$$

PROPOSAL. *Take the coefficients N^k_n according to the expansion*

$$N^k_n = -l^k\frac{\partial F}{\partial x^n} - Fm^k\breve{P}_n \tag{2.6}$$

with

$$\breve{P}_n = \frac{\partial\theta}{\partial x^n} - k_n, \tag{2.7}$$

where $k_n = k_n(x)$ is a covariant vector field, such that the equalities

$$d_nF = 0, \qquad d_n\theta = k_n, \qquad y_kN^k_{nmi} = 0 \tag{2.8}$$

be realized.

The vanishing $d_nF = 0$ and the equality $d_n\theta = k_n$ just follow from the choice (2.6). Considering two values $\theta_1 = \theta(x,y_1)$ and $\theta_2 = \theta(x,y_2)$, we have

$$d_n\theta_1 = \frac{\partial\theta_1}{\partial x^n} + N^k_n(x,y_1)\frac{\partial\theta_1}{\partial y^k_1}, \qquad d_n\theta_2 = \frac{\partial\theta_2}{\partial x^n} + N^k_n(x,y_2)\frac{\partial\theta_2}{\partial y^k_2}, \tag{2.9}$$

and from $d_n\theta = k_n$ we may conclude that *the preservation*

$$d_n(\theta_2 - \theta_1) = 0 \tag{2.10}$$

holds because the vector field k_n is independent of tangent vectors y.

From (2.6) it follows directly that

$$N^k{}_{nm} = -l^k\frac{\partial l_m}{\partial x^n} - l_mm^k\breve{P}_n - F\frac{\partial m^k}{\partial y^m}\breve{P}_n - m^k\frac{\partial m_m}{\partial x^n}. \tag{2.11}$$

It is convenient to use the identities

$$F\frac{\partial m_k}{\partial y^m}=-l_km_m+Im_mm_k, \qquad F\frac{\partial m^k}{\partial y^m}=-Im^km_m-l^km_m \tag{2.12}$$

(they are tantamount to the identities written in formula (6.22) of chapter 6 in [1]), together with their immediate implication

$$F\frac{\partial(m^km_n)}{\partial y^m}=-(l_nm^km_m+l^km_nm_m).$$

Short evaluations show that

$$N^k{}_{nmi}=\frac{1}{F}m^km_m\left(F\frac{\partial I}{\partial y^i}\breve{P}_n-m_i\frac{\partial I}{\partial x^n}\right). \tag{2.13}$$

Indeed, from (2.11) it follows straightforwardly that

$$N^k{}_{nmi}=-\frac{1}{F}h^k_i\partial_nl_m-l^k\partial_n\left(\frac{1}{F}h_{mi}\right)-\frac{1}{F}h_{mi}m^k\breve{P}_n+l_m\left(I\,m^km_i+l^km_i\right)\breve{P}_n-l_mm^k\partial_n\left(\frac{1}{F}m_i\right)$$

$$+\frac{\partial I}{\partial y^i}\,m^km_m\breve{P}_n+I\,m^km_m\partial_n\left(\frac{1}{F}m_i\right)-Il^km_im_m\breve{P}_n-Im^kl_mm_i\breve{P}_n$$

$$+\frac{1}{F}h^k_im_m\breve{P}_n-l^k(l_m-Im_m)m_i\breve{P}_n+l^km_m\partial_n\left(\frac{1}{F}m_i\right)$$

$$+(Im^k+l^k)m_i\frac{1}{F}\partial_nm_m+m^k\partial_n\left(\frac{1}{F}l_mm_i-\frac{1}{F}I\,m_im_m\right),$$

where ∂_n means $\partial/\partial x^n$. The identity $h_{mi}=m_mm_i$ has been taken into account. Canceling similar terms leads to (2.13).

Owing to the identity $l_km^k=0$, the vanishing $l_kN^k{}_{nmi}=0$ holds fine.

Because of $y^i\partial_{y^i}I=0$, the equality (2.13) can be written in the concise form

$$N^k{}_{nmi}=-\frac{1}{F}m^km_mm_id_nI\equiv-\frac{1}{F}A^k{}_{mi}d_n\ln|I|. \tag{2.14}$$

The sought *Finsler connection*

$$\mathcal{FC}=\{N^k_n,D^k{}_{nm}\} \tag{2.15}$$

involves also the coefficients $D^k{}_{nm}=D^k{}_{nm}(x,y)$ which are required to construct the operator of the covariant derivative $\mathcal{D}_n$ which action is exemplified in the conventional way:

$$\mathcal{D}_nw^k{}_m\ :=d_nw^k{}_m+D^k{}_{nh}w^h{}_m-D^h{}_{nm}w^k{}_h, \tag{2.16}$$

where $w^k{}_m=w^k{}_m(x,y)$ is an arbitrary differentiable (1,1)-type tensor.

If we differentiate the vanishing $d_iF=0$ with respect to the variable y^j and multiply the result by F, we obtain the vanishing

$$\mathcal{D}_iy_j\ :=\frac{\partial y_j}{\partial x^i}+N^k_ig_{kj}-D^h{}_{ij}y_h=0 \tag{2.17}$$

when the choice

$$D^k{}_{in} = -N^k{}_{in} \tag{2.18}$$

is made. Differentiating (2.17) with respect to y^n just manifests that the choice is also of success to fulfill the metricity condition

$$\mathcal{D}_i g_{jn} \ := d_i g_{jn} - D^h{}_{ij} g_{hn} - D^h{}_{in} g_{jh} = 0, \tag{2.19}$$

because of $y_k N^k_{nmi} = 0$.

If we contract (2.18) by y^n and take into account the definition of the coefficients $N^k{}_{in}$ indicated in (2.5), we obtain the equality

$$N^k_i = -D^k{}_{in} y^n. \tag{2.20}$$

The contravariant version of the vanishing (2.17) is obtained through the chain

$$\mathcal{D}_i y^j \ := d_i y^j + D^j{}_{ih} y^h = N^j_i + D^j{}_{ih} y^h = 0. \tag{2.21}$$

Because of $\mathcal{D}_i h^{nk} = 0$, applying the derivative $\mathcal{D}_i$ to the equality $h^{nk} = m^n m^k$ (see (2.2)) and contracting the result by m_n, we conclude that

$$\mathcal{D}_i m^k = 0, \quad \text{which means} \ \ d_i m^k = N^k{}_{ih} m^h. \tag{2.22}$$

Because of the homogeneity, the unit tangent vector components $l^n = l^n(x, y)$ can obviously be regarded as functions $l^n = L^n(x, \theta)$ of the pair (x, θ). Let us denote $l^n_\theta = \partial L^n/\partial\theta$. Since $\partial F/\partial\theta = 0$ and $l_n l^n_\theta = 0$, we may conclude from (2.3) that $l^n_\theta = m^n$. Measuring the length of the indicatrix (which is defined by $F = 1$) by means of a parameter s, so that $ds = \sqrt{g_{ij} dl^i dl^j}$, we obtain $ds = \sqrt{g_{ij} l^i_\theta l^j_\theta}\, d\theta = d\theta$, assuming $ds > 0$ and $d\theta > 0$. Thus

$$ds = d\theta \quad \text{along the indicatrix}, \tag{2.23}$$

which explains us that the θ *measures the length of indicatrix.*

If at a fixed x we introduce in the tangent space $T_x M$ the coordinates $z^A = \{z^1 = F, z^2 = \theta\}$ and consider the respective transforms

$$G_{AB} = g_{ij} \frac{\partial y^i}{\partial z^A} \frac{\partial y^i}{\partial z^B}, \quad A = 1, 2, \ B = 1, 2, \tag{2.24}$$

of the Finsler metric tensor components g_{ij}, we obtain simply

$$G_{11} = 1, \quad G_{12} = 0, \quad G_{22} = F^2. \tag{2.25}$$

With these components, it is easy to calculate the area of domain of the tangent space $T_x M$ by using the integral measure

$$\int \sqrt{\det(G_{AB})}\, dz^1 dz^2 = \int F dF d\theta. \tag{2.26}$$

In particular, for the sector $\sigma_{\{y_1, y_2\}} \subset T_x M$ bounded by the vectors y_1, y_2 and the indicatrix arc we obtain by integration the area $\Sigma_{\{y_1, y_2\}}$ which is given by

$$\Sigma_{\{y_1, y_2\}} = \frac{1}{2}(\theta_2 - \theta_1), \tag{2.27}$$

so that in the two-dimensional case the angle in the Finsler geometry can be defined by the area just in the same way as in the Riemannian geometry. The difference

$$\theta_2 - \theta_1 = \theta(x, y_2) - \theta(x, y_1) \tag{2.28}$$

can naturally be regarded as the value of angle between two vectors $y_1, y_2 \in T_xM$, the *two-vector angle* for short. The formula (2.27) is equivalent to Definition (2) of [6] which was proposed to define angle by area; we use the right orientation of angle.

As well as the area is attributed to the tangent space by means of the integral measure (2.26) and the conditions $d_nF = 0$ and $d_n(\theta_2 - \theta_1) = 0$ are fulfilled, *the angle-preserving connection keeps the area under parallel transports* along curves joining point to point in the background manifold. Thus we are entitled to set forth the thesis: *the angle-preserving connection is the area-preserving connection.*

3. Curvature tensor

With arbitrary coefficients $\{N^k_n, D^k{}_{nm}\}$, commuting the covariant derivative (2.16) yields the equality

$$(\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i)\, w^n{}_k = M^h{}_{ij}\frac{\partial w^n{}_k}{\partial y^h} - E_k{}^h{}_{ij}w^n{}_h + E_h{}^n{}_{ij}w^h{}_k \tag{3.1}$$

with the tensors

$$M^n{}_{ij} := d_iN^n_j - d_jN^n_i \tag{3.2}$$

and

$$E_k{}^n{}_{ij} := d_iD^n{}_{jk} - d_jD^n{}_{ik} + D^m{}_{jk}D^n{}_{im} - D^m{}_{ik}D^n{}_{jm}. \tag{3.3}$$

If the choice $D^k{}_{in} = -N^k{}_{in}$ is made (see (2.18)), the tensor (3.2) can be written in the form

$$M^n{}_{ij} = \frac{\partial N^n_j}{\partial x^i} - \frac{\partial N^n_i}{\partial x^j} - N^h_iD^n{}_{jh} + N^h_jD^n{}_{ih}, \tag{3.4}$$

which entails the equality

$$E_k{}^n{}_{ij} = -\frac{\partial M^n{}_{ij}}{\partial y^k}. \tag{3.5}$$

By applying the commutation rule (3.1) to the vanishing set $\{\mathcal{D}_iF = d_iF = 0, \mathcal{D}_iy^n = 0, \mathcal{D}_iy_k = 0, \mathcal{D}_ig_{nk} = 0\}$, we respectively obtain the identities

$$y_nM^n{}_{ij} = 0, \quad y^kE_k{}^n{}_{ij} = -M^n{}_{ij}, \quad y_nE_k{}^n{}_{ij} = g_{kn}M^n{}_{ij}, \quad E_{mnij} + E_{nmij} = 2C_{mnh}M^h{}_{ij}. \tag{3.6}$$

In case of the coefficients $\{N^k_n, D^k{}_{nm}\}$ proposed by (2.6) and (2.18) the direct calculation of the right-hand parts in (3.2) and (3.3) results in

Theorem 3.1. *The tensors $M^n{}_{ij}$ and $E_k{}^n{}_{ij}$ are represented by the following simple and explicit formulas:*

$$M^n{}_{ij} = Fm^nM_{ij} \tag{3.7}$$

and

$$E_k{}^n{}_{ij} = \Big(-l_km^n + l^nm_k + Im_km^n\Big)M_{ij}, \tag{3.8}$$

where

$$M_{ij}=\frac{\partial k_j}{\partial x^i}-\frac{\partial k_i}{\partial x^j}. \tag{3.9}$$

It proves pertinent to replace in the commutator (3.1) the partial derivative $\partial w^n{}_k/\partial y^h$ by the definition

$$\mathcal{S}_h w^n{}_k=\frac{\partial w^n{}_k}{\partial y^h}+C^n{}_{hk}w^h{}_k-C^m{}_{hk}w^n{}_m \tag{3.10}$$

which has the meaning of the covariant derivative in the tangent Riemannian space $\mathcal{R}_{\{x\}}$. With the *curvature tensor*

$$\rho_k{}^n{}_{ij}=E_k{}^n{}_{ij}-M^h{}_{ij}C^n{}_{hk}, \tag{3.11}$$

the commutator takes on the form

$$(\mathcal{D}_i\mathcal{D}_j-\mathcal{D}_j\mathcal{D}_i)\,w^n{}_k=M^h{}_{ij}\mathcal{S}_h w^n{}_k-\rho_k{}^h{}_{ij}w^n{}_h+\rho_h{}^n{}_{ij}w^h{}_k. \tag{3.12}$$

The skew-symmetry

$$\rho_{mnij}=-\rho_{nmij} \tag{3.13}$$

holds (cf. the last item in (3.6)).

If we take into account the form of the tensor C_{ijk} indicated in (2.2), from (3.8) and (3.11) we may conclude that the curvature tensor is of the following astonishingly simple form:

$$\rho_k{}^n{}_{ij}=(l^n m_k-l_k m^n)M_{ij}\equiv\epsilon^n{}_k M_{ij}. \tag{3.14}$$

The tensor $\rho_{knij}=g_{nm}\rho_k{}^m{}_{ij}$ can be represented in the form

$$\rho_{knij}=\epsilon_{nk}M_{ij}. \tag{3.15}$$

We have $l^k m_n\rho_k{}^n{}_{ij}=-M_{ij}$.

4. Riemannian counterparts

If the Finsler space $\mathcal{F}_2$ is a Riemannian space with a Riemannian metric function $S=\sqrt{a_{ij}y^iy^j}$, where $a_{ij}=a_{ij}(x)$ is a Riemannian metric tensor, we can consider the Riemannian precursor coefficients

$$L^k_n=N^k_n\Big|_{\text{Riemannian limit}}. \tag{4.1}$$

From (2.6) it follows that

$$L^k_n=-y^k\frac{1}{S}\frac{\partial S}{\partial x^n}-Sm^k\left(\frac{\partial\theta}{\partial x^n}-k_n\right). \tag{4.2}$$

On the other hand, denoting by $a^k{}_{nh}$ the Riemannian Christoffel symbols constructed from the Riemannian metric tensor a_{mn}, we can obtain the equality

$$a^k{}_{nh}y^h=\frac{1}{S}\frac{\partial S}{\partial x^n}y^k+\left(\frac{\partial\theta}{\partial x^n}+n^h\nabla_n\widetilde{b}_h\right)Sm^k \tag{4.3}$$

(see (A.9) in Appendix A), where $\widetilde{b}_h = \widetilde{b}_h(x)$ is a vector field chosen to fulfill

$$\theta(x, \widetilde{b}(x)) = 0 \tag{4.4}$$

and the pair $n_i, \widetilde{b}_i$ is orthonormal with respect to the tensor a_{ij}. The reciprocal pair is $\{\widetilde{b}^i, n^i\}$ with $\widetilde{b}^i = a^{ij}\widetilde{b}_j$ and $n^i = a^{ij}n_j$, where a^{ij} is the inverse of a_{ij}. The ∇_n stands for the Riemannian covariant derivative taken with $a^k{}_{nh}$. We get

$$L^k_n = Sm^k T_n - a^k{}_{nh} y^h \equiv L^k{}_{nh} y^h \tag{4.5}$$

with

$$L^k{}_{nh} = -a^{kj}\epsilon^{\mathrm{Riem}}_{jh} T_n - a^k{}_{nh} \tag{4.6}$$

and

$$T_n = n^h \nabla_n \widetilde{b}_h + k_n. \tag{4.7}$$

$\epsilon^{\mathrm{Riem}}_{jh} = \sqrt{\det(a_{mn})}\,\gamma_{jh}$, where $\gamma_{11} = \gamma_{22} = 0$ and $\gamma_{12} = -\gamma_{21} = 1$. The metricity property

$$\frac{\partial a_{mn}}{\partial x^i} + L^s{}_{im} a_{sn} + L^s{}_{in} a_{ms} = 0 \tag{4.8}$$

holds independently of presence of the vector T_n. In contrast to the Christoffel symbols $a^k{}_{nh}$, the coefficients $L^k{}_{nh}$ obtained are not symmetric with respect to the subscripts.

Let us take the coefficients $\bar{L}^n{}_{ik} = -L^n{}_{ik}$ from (4.6) and construct the tensor

$$\bar{L}_k{}^n{}_{ij} = \frac{\partial \bar{L}^n{}_{jk}}{\partial x^i} - \frac{\partial \bar{L}^n{}_{ik}}{\partial x^j} + \bar{L}^m{}_{jk}\bar{L}^n{}_{im} - \bar{L}^m{}_{ik}\bar{L}^n{}_{jm} \equiv \bar{L}_k{}^n{}_{ij}(x). \tag{4.9}$$

We obtain

$$\bar{L}_k{}^n{}_{ij} = (\nabla_i T_j - \nabla_j T_i)\, a^{nt}\epsilon^{\mathrm{Riem}}_{tk} + a_k{}^n{}_{ij}, \tag{4.10}$$

where

$$a_k{}^n{}_{ij} = \frac{\partial a^n{}_{jk}}{\partial x^i} - \frac{\partial a^n{}_{ik}}{\partial x^j} + a^m{}_{jk} a^n{}_{im} - a^m{}_{ik} a^n{}_{jm} \tag{4.11}$$

is the Riemannian curvature tensor constructed from the Riemannian metric tensor a_{mn}. We have taken into account the vanishing $\nabla_i \epsilon^{\mathrm{Riem}}_{tk} = 0$.

From the equalities

$$\nabla_i \widetilde{b}^k = -n^k \widetilde{b}_m \nabla_i n^m, \qquad \nabla_i n^k = -\widetilde{b}^k n_m \nabla_i \widetilde{b}^m \tag{4.12}$$

it follows that

$$\nabla_i(n^t \nabla_j \widetilde{b}_t) - \nabla_j(n^t \nabla_i \widetilde{b}_t) = n^t(\nabla_i \nabla_j \widetilde{b}_t - \nabla_j \nabla_i \widetilde{b}_t) = -n^t \widetilde{b}_l a_t{}^l{}_{ij}. \tag{4.13}$$

Therefore, taking the T_i from (4.7), we find that

$$\nabla_i T_j - \nabla_j T_i = M_{ij} - n^t \widetilde{b}_l a_t{}^l{}_{ij}.$$

Noting the equality

$$n^t \widetilde{b}_l a_t{}^l{}_{ij} a^{nt} \epsilon^{\mathrm{Riem}}_{tk} = a_k{}^n{}_{ij} \tag{4.14}$$

(see Appendix A), we conclude that the tensor (4.10) can be read merely

$$\bar{L}_k{}^n{}_{ij} = a^{nt}\epsilon^{\mathrm{Riem}}_{tk} M_{ij}. \tag{4.15}$$

If we want to have $\bar{L}^s{}_{ij} = a^s{}_{ij}$, we must make the choice

$$k_n = -n^h \nabla_n \widetilde{b}_h, \tag{4.16}$$

which entails $T_i = 0$, in which case the tensor $\bar{L}_k{}^n{}_{ij}$ given by (4.9) is the ordinary Riemannian curvature tensor $a_k{}^n{}_{ij}$.

If the Finsler space $\mathcal{F}_2$ is not a Riemannian space, it is possible to introduce the *θ-associated Riemannian space* $\mathcal{R}_{\{\theta\}}$ as follows.

The angle function $\theta = \theta(x, y)$ is defined (from equation (2.3)) up to an arbitrary integration constant which may depend on x, which is the reason why $d_n\theta$ should not be put to be zero in (2.8) (in distinction from the vanishing $d_nF = 0$). There exists the freedom to make the redefinition $\theta \to \theta + C(x)$. To specify the value of θ unambiguously in a fixed tangent space T_xM, we need in this T_xM an axis from which the value is to be measured. Let the distribution of these axes over the base manifold be assigned by means of a contravariant vector field $b^i = b^i(x)$. Then we obtain precisely the equality $\theta(x, b(x)) = 0$ which does not permit the redefinitions anymore.

It is appropriate to construct the *osculating* Riemannian metric tensor $a_{mn}(x) = g_{mn}(x, b(x))$ and introduce the normalized vector $\widetilde{b}^i = b^i/\sqrt{a_{mn}b^m b^n}$. Because of the homogeneity, $g_{mn}(x, b(x)) = g_{mn}\big(x, \widetilde{b}(x)\big)$. The vector $n_i(x)$ can be taken to equal the value of the derivative $\partial\theta/\partial y^i$ at the argument pair $\big(x, \widetilde{b}(x)\big)$. Then, because of $\theta(x, \widetilde{b}(x)) = 0$ and $g_{ij} = l_i l_j + m_i m_j$ (see (2.2) and (2.3)), the pair $\{\widetilde{b}_i, n_i\}$ thus introduced is orthonormal with respect to the tensor a_{ij} produced by osculation. This tensor a_{ij} introduce the Riemannian space $\mathcal{R}_{\{\theta\}}$ on the base manifold M. We obtain the equalities

$$a_{mn}\widetilde{b}^m\widetilde{b}^n = 1, \qquad a_{mn}n^m n^n = 1, \qquad a_{mn}\widetilde{b}^m n^n = 0, \tag{4.17}$$

and $F\big(x, \widetilde{b}(x)\big) = 1$ together with

$$a_{mn}(x) = g_{mn}\big(x, \widetilde{b}(x)\big), \quad \theta(x, \widetilde{b}(x)) = 0, \quad \frac{\partial\theta}{\partial y^i}\big(x, \widetilde{b}(x)\big) = n_i(x), \quad \frac{\partial\theta}{\partial y^i}\big(x, n(x)\big) = -\widetilde{b}_i(x). \tag{4.18}$$

The arisen expansion $y^m = \widetilde{b}\widetilde{b}^m + nn^m$ is convenient to use in many fragments of evaluations. The last equality in the list (4.18) is explicated from (2.1).

Now we can download in the space $\mathcal{R}_{\{\theta\}}$ all the relations (4.2)-(4.16) formulated above in the Riemannian precursor space. On doing so, we can conclude after comparing (4.15) with (3.15) that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij}$ is factorable, namely

$$\rho_{knij} = f_1 \bar{L}_{knij} \quad \text{with} \quad \bar{L}_{knij} = a_{nh}\bar{L}_k{}^h{}_{ij} \equiv \bar{L}_{knij}(x), \tag{4.19}$$

where

$$f_1 = \sqrt{\frac{\det(g_{hl})}{\det(a_{mn})}}. \tag{4.20}$$

We have arrived at the following theorem.

Theorem 4.1. *The curvature tensor $\rho_k{}^n{}_{ij}$ of the Finsler space $\mathcal{F}_2$ equipped with the angle-preserving connection is such that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij} = \rho_{knij}(x, y)$*

is proportional to the tensor $\bar{L}_{knij} = a_{nh}\bar{L}_k{}^h{}_{ij} = \bar{L}_{knij}(x)$ *which does not involve any dependence on tangent vectors. The factor of proportionality* f_1 *is expressed through the determinants of metric tensors, according to* (4.20).

5. Conclusions

In the Riemannian geometry the contraction $a^k{}_{nh}y^h$ of the Christoffel symbols $a^k{}_{nh}$ with the tangent vector y admits the angle representation (4.3)–(4.4):

$$a^k{}_{nh}y^h = \frac{1}{S}y^k\frac{\partial S}{\partial x^n} + \left(\frac{\partial \theta}{\partial x^n} - t_n(x)\right) Sm^k, \tag{5.1}$$

where $t_n(x) = -n^h\nabla_n\widetilde{b}_h$ and $\theta(x, \widetilde{b}(x)) = 0$. Why don't lift the representation to the Finsler level to take the coefficients $N^k_n = N^k_n(x, y)$ in the operator $d_n = \partial_{x^n} + N^k_n(x, y)\partial_{y^k}$ to be of the similar form? Our proposal in (2.6) was of this kind, namely,

$$N^k_n = -\frac{1}{F}y^k\frac{\partial F}{\partial x^n} - \left(\frac{\partial \theta}{\partial x^n} - k_n(x)\right) Fm^k. \tag{5.2}$$

At any k_n, the vanishing $d_nF = d_n\theta - k_n = d_n(\theta_2 - \theta_1) = 0$ immediately ensues from this proposal. It is a big (and good) surprise that the vanishing $y_kN^k{}_{nmi} = 0$ ensues also, which enables us to obtain the covariant derivative $\mathcal{D}_n$ possessing the metric property $\mathcal{D}_ng_{ij} = 0$, where $\mathcal{D}_ng_{ij} = d_ng_{ij} - D^h{}_{ni}g_{hj} - D^h{}_{nj}g_{ih}$ with the connection coefficients $D^h{}_{ni} = -N^h{}_{ni}$. If we want to obtain torsionless coefficients in the Riemannian limit of these $D^h{}_{ni}$, we should take the vector field $k_n(x)$ to be the field $t_n(x) = -n^h\nabla_n\widetilde{b}_h$ which enters the Riemannian version (5.1).

The induced parallel transports of the objects $\{F, \theta_2 - \theta_1, g_{ij}\}$ along the horizontal curves (running on the base manifold M) are represented infinitesimally by the elements $\{dx^nd_nF, dx^nd_n(\theta_2 - \theta_1), dx^n\mathcal{D}_ng_{ij}\}$ which all are the naught because of $d_nF = d_n(\theta_2 - \theta_1) = \mathcal{D}_ng_{ij} = 0$. Therefore the transports realize isometries of the tangent Riemannian spaces $\mathcal{R}_{\{x\}}$ supported by points $x \in M$, taking indicatrices into indicatrices. The coefficients N^k_n given by (2.6) are in general non-linear with respect to the variable y.

In the Riemannian case, the right-hand part of (5.1) can be expressed through the Christoffel symbols and, therefore, can be constructed from the first derivatives of the metric tensor. This is the privilege of the Riemannian geometry which lives in the ground floor of the Finsler building, — the right-hand part of the Finsler coefficients (5.2) is not a composition of partial derivatives of the Finsler metric tensor. In distinction to the Riemannian geometry which provides us with simple and explicit angle (see (A.8) in Appendix A), the Finsler angle function $\theta = \theta(x, y)$ is defined by the partial differentiable equation (2.3) which cannot be integrated explicitly, except for rare particular cases of the Finsler metric function.

The second big surprise is that the angle-preserving connection obtained in this way admits the C^∞-regular realization globally regarding the dependence on tangent vectors. Such a realization takes place for the Finsleroid-regular metric function, the Randers metric function, and probably for many other Finsler metric functions.

The Finsler connection obtained does not need any facility which could be provided by the geodesic spray coefficients. Due attention to the angle wisdom is sufficient: the Finsler space is connected by its angle structure, similarly to the well known property of Riemannian geometry.

Our consideration was restricted by the dimension 2. Development of due extensions to higher dimensions is the problem of urgent kind.

Appendix A: Involved evaluations

Let us verify theorem 3.1. Using $d_i m^k = N^k{}_{ih} m^h$ (see (2.22)) and (2.11), we obtain

$$d_i m^k = -m^h l^k \frac{\partial l_h}{\partial x^i} + (Im^k + l^k)\breve{P}_i - m^h m^k \frac{\partial m_h}{\partial x^i}.$$

Also,

$$d_i \frac{\partial \theta}{\partial x^j} - d_j \frac{\partial \theta}{\partial x^i} = N_i^t \frac{\partial \frac{1}{F} m_t}{\partial x^j} - N_j^t \frac{\partial \frac{1}{F} m_t}{\partial x^i}$$

$$= N_i^t \left[-\frac{1}{F^2} \frac{\partial F}{\partial x^j} m_t + \frac{1}{F} \frac{\partial m_t}{\partial x^j} \right] - N_j^t \left[-\frac{1}{F^2} \frac{\partial F}{\partial x^i} m_t + \frac{1}{F} \frac{\partial m_t}{\partial x^i} \right]$$

$$= m^t \breve{P}_i \left[\frac{1}{F} \frac{\partial F}{\partial x^j} m_t - \frac{\partial m_t}{\partial x^j} \right] - m^t \breve{P}_j \left[\frac{1}{F} \frac{\partial F}{\partial x^i} m_t - \frac{\partial m_t}{\partial x^i} \right].$$

Using these formulas in (3.2) and keeping also in mind that $d_i F = 0$ and $d_i y^k = N_i^k$, from (2.6) we get

$$M^n{}_{ij} = -N_i^n \frac{1}{F} \frac{\partial F}{\partial x^j} + N_j^n \frac{1}{F} \frac{\partial F}{\partial x^i} - l^n N_i^h \frac{\partial \frac{\partial F}{\partial x^j}}{\partial y^h} + l^n N_j^h \frac{\partial \frac{\partial F}{\partial x^i}}{\partial y^h}$$

$$+ m^h y^n \left(\breve{P}_j \frac{\partial l_h}{\partial x^i} - \breve{P}_i \frac{\partial l_h}{\partial x^j} \right) + F m^h m^n \left(\breve{P}_j \frac{\partial m_h}{\partial x^i} - \breve{P}_i \frac{\partial m_h}{\partial x^j} \right)$$

$$- F m^n \left(m^t \breve{P}_i \left(\frac{1}{F} \frac{\partial F}{\partial x^j} m_t - \frac{\partial m_t}{\partial x^j} \right) - m^t \breve{P}_j \left(\frac{1}{F} \frac{\partial F}{\partial x^i} m_t - \frac{\partial m_t}{\partial x^i} \right) \right) + F m^n M_{ij},$$

which reduces to

$$M^n{}_{ij} = -l^n N_i^h \frac{\partial \frac{\partial F}{\partial x^j}}{\partial y^h} + l^n N_j^h \frac{\partial \frac{\partial F}{\partial x^i}}{\partial y^h} + m^h y^n \left(\breve{P}_j \frac{\partial l_h}{\partial x^i} - \breve{P}_i \frac{\partial l_h}{\partial x^j} \right) + F m^n M_{ij} = F m^n M_{ij}.$$

The representation (3.7) holds. Differentiating (3.7) with respect of y^k and taking into account the formula (3.5) together with (2.12), the representation (3.8) is obtained. The theorem is valid.

Let us verify the equality (4.14). Since the dimension is $N=2$, the Riemannian curvature tensor $a_{tlin}=a_{lh}a_t{}^h{}_{in}$ possesses the representation

$$a_{tlij}=-R(a_{tj}a_{li}-a_{ti}a_{lj}). \tag{A.1}$$

The meaning of the scalar R is explained by the equality

$$a^{ti}a^{lj}a_{tlij}=2R. \tag{A.2}$$

Inserting the expansion $a_{tn}=\widetilde{b}_t\widetilde{b}_n+n_tn_n$ in (A.1) results in the factorization

$$a_{tlij}=-R(n_t\widetilde{b}_l-n_l\widetilde{b}_t)(n_j\widetilde{b}_i-n_i\widetilde{b}_j). \tag{A.3}$$

It follows that $n^t\widetilde{b}^la_{tlij}=-R(n_j\widetilde{b}_i-n_i\widetilde{b}_j)$ and therefore

$$n^t\widetilde{b}^la_{tlij}(\widetilde{b}_kn_n-n_k\widetilde{b}_n)=-R(n_j\widetilde{b}_i-n_i\widetilde{b}_j)(\widetilde{b}_kn_n-n_k\widetilde{b}_n)=-a_{knij}. \tag{A.4}$$

We can substitute here $\epsilon^{\text{Riem}}_{kn}$ with $\widetilde{b}_kn_n-n_k\widetilde{b}_n$. The equality (4.14) is valid.

In the remainder we verify the representation of $a^k{}_{nh}y^h$ indicated in (2.25).

In the Riemannian limit we have $||b||=1$ and may use the notation b_i instead of $\widetilde{b}_i$. We start with the Riemannian representations

$$a_{ij}=n_in_j+b_ib_j,\qquad l^i=\frac{y^i}{S},\qquad l_i=a_{ij}l^j,\qquad S=\sqrt{a_{mn}y^my^n},$$

$$l^i=n^i\sin\theta+b^i\cos\theta,\qquad l_i=n_i\sin\theta+b_i\cos\theta,$$

and

$$m^i=n^i\cos\theta-b^i\sin\theta,\qquad m_i=n_i\cos\theta-b_i\sin\theta.$$

We have

$$n=S\sin\theta,\qquad b=S\cos\theta,$$

which entails

$$m^i=\frac{1}{S}(bn^i-nb^i),\quad m_i=\frac{1}{S}(bn_i-nb_i),\quad l^il_i=m^im_i=1,\quad l^im_i=0. \tag{A.5}$$

We find

$$a_{ij}-l_il_j=n_in_j+b_ib_j-n_i\sin\theta(n_j\sin\theta+b_j\cos\theta)-b_i\cos\theta(n_j\sin\theta+b_j\cos\theta)$$

$$=n_in_j\cos^2\theta+b_ib_j\sin^2\theta-(n_ib_j+n_jb_i)\sin\theta\cos\theta,$$

or

$$a_{ij}-l_il_j=(n_i\cos\theta-b_i\sin\theta)(n_j\cos\theta-b_j\sin\theta)\equiv m_im_j.$$

Noting that

$$\frac{\partial l_i}{\partial x^n}=\frac{1}{S}(nn_{i,n}+bb_{i,n}+m_i\,\theta_{,n}),\quad \frac{\partial m_i}{\partial x^n}=\frac{1}{S}(bn_{i,n}-nb_{i,n}-y_i\theta_{,n})$$

(by $\{,n\}$ we denote the partial derivative with respect to x^n), from the expansion $a_{ij} = l_i l_j + m_i m_j$ we obtain

$$a_{ij,n} = \frac{1}{S}(nn_{i,n} + bb_{i,n} + Sm_i\,\theta_{,n})l_j + \frac{1}{S}(nn_{j,n} + bb_{j,n} + Sm_j\,\theta_{,n})l_i$$

$$+\frac{1}{S}(bn_{i,n} - nb_{i,n} - y_i\theta_{,n})m_j + \frac{1}{S}(bn_{j,n} - nb_{j,n} - y_j\theta_{,n})m_i,$$

which can be written in the simpler form

$$a_{ij,n} = \frac{1}{S}(nn_{i,n}+bb_{i,n})l_j+\frac{1}{S}(nn_{j,n}+bb_{j,n})l_i+\frac{1}{S}(bn_{i,n}-nb_{i,n})m_j+\frac{1}{S}(bn_{j,n}-nb_{j,n})m_i. \quad \text{(A.6)}$$

Let us compose the sum

$$a_{in,j} + a_{jn,i} - a_{ij,n} =$$

$$\frac{1}{S}(nn_{i,j} + bb_{i,j})l_n + \frac{1}{S}(nn_{n,j} + bb_{n,j})l_i + \frac{1}{S}(bn_{i,j} - nb_{i,j})m_n + \frac{1}{S}(bn_{n,j} - nb_{n,j})m_i$$

$$+\frac{1}{S}(nn_{j,i} + bb_{j,i})l_n + \frac{1}{S}(nn_{n,i} + bb_{n,i})l_j + \frac{1}{S}(bn_{j,i} - nb_{j,i})m_n + \frac{1}{S}(bn_{n,i} - nb_{n,i})m_j$$

$$-\frac{1}{S}(nn_{i,n} + bb_{i,n})l_j - \frac{1}{S}(nn_{j,n} + bb_{j,n})l_i - \frac{1}{S}(bn_{i,n} - nb_{i,n})m_j - \frac{1}{S}(bn_{j,n} - nb_{j,n})m_i,$$

from which we obtain

$$(a_{in,j} + a_{jn,i} - a_{ij,n})y^j =$$

$$= \frac{1}{S}(nn_{i,j} + bb_{i,j})y^j l_n + \frac{1}{S}(nn_{n,j} + bb_{n,j})y^j l_i + \frac{1}{S}(bn_{i,j} - nb_{i,j})y^j m_n + \frac{1}{S}(bn_{n,j} - nb_{n,j})y^j m_i$$

$$+\frac{1}{S}(nn_{j,i} + bb_{j,i})y^j l_n + (nn_{n,i} + bb_{n,i}) + \frac{1}{S}(bn_{j,i} - nb_{j,i})y^j m_n$$

$$-(nn_{i,n} + bb_{i,n}) - \frac{1}{S}(nn_{j,n} + bb_{j,n})y^j l_i - \frac{1}{S}(bn_{j,n} - nb_{j,n})y^j m_i,$$

or

$$(a_{in,j} + a_{jn,i} - a_{ij,n})y^j = \frac{1}{S}(nn_{i,j} + bb_{i,j})y^j l_n + \frac{1}{S}\Big(n(n_{n,j} - n_{j,n}) + b(b_{n,j} - b_{j,n})\Big)y^j l_i$$

$$+\frac{1}{S}(bn_{i,j} - nb_{i,j})y^j m_n + \frac{1}{S}\Big(b(n_{n,j} - n_{j,n}) - n(b_{n,j} - b_{j,n})\Big)y^j m_i$$

$$+\frac{1}{S}(nn_{j,i}+bb_{j,i})y^jl_n+n(n_{n,i}-n_{i,n})+b(b_{n,i}-b_{i,n})+\frac{1}{S}(bn_{j,i}-nb_{j,i})y^jm_n. \tag{A.7}$$

We use here the equalities

$$\nabla_n b_k=p_nn_k, \qquad \nabla_n n_k=-p_nb_k, \qquad \text{with } \; p_n=n^h\nabla_n b_h,$$

where ∇_n stands for the Riemannian covariant derivative taken with the Riemannian Christoffel symbols $a^k{}_{nh}$ constructed from the Riemannian metric tensor a_{mn}, which entails

$$n(n_{n,j}-n_{j,n})+b(b_{n,j}-b_{j,n})=S(p_jm_n-p_nm_j).$$

The method yields

$$(a_{in,j}+a_{jn,i}-a_{ij,n})y^j=\frac{1}{S}(nn_{i,j}+bb_{i,j})y^jl_n+y^jp_jm_nl_i$$

$$+\frac{1}{S}(bn_{i,j}-nb_{i,j})y^jm_n+\Big[b(-p_jb_n+p_nb_j)-n(p_jn_n-p_nn_j)\Big]l^jm_i$$

$$+\frac{1}{S}(nn_{j,i}+bb_{j,i})y^jl_n+S(p_im_n-p_nm_i)+\frac{1}{S}(bn_{,i}-nb_{,i})m_n$$

$$=\frac{1}{S}\Big(n(n_{i,j}+n_{j,i})+b(b_{i,j}+b_{j,i})\Big)y^jl_n+y^jp_jm_nl_i$$

$$+\frac{1}{S}\Big(b(-p_jb_i+p_ib_j)-n(p_jn_i-p_in_j)\Big)y^jm_n+Sp_im_n+\frac{2}{S}(bn_{,i}-nb_{,i})m_n-y^jp_jl_nm_i$$

$$=\frac{1}{S}\Big(2SS_{,i}+n(-p_jb_i+p_ib_j)y^j+b(p_jn_i-p_in_j)y^j\Big)l_n+2Sp_im_n+\frac{2}{S}(bn_{,i}-nb_{,i})m_n-y^jp_jl_nm_i.$$

We obtain eventually the expansion

$$\frac{1}{2}(a_{in,j}+a_{jn,i}-a_{ij,n})y^j=S_{,i}l_n+\Big(p_i+\frac{1}{S^2}(bn_{,i}-nb_{,i})\Big)Sm_n.$$

Since

$$\theta=\arctan\frac{n}{b}, \qquad \theta_{,n}=\frac{1}{S^2}(bn_{,n}-nb_{,n}), \tag{A.8}$$

we get

$$\frac{1}{2}(a_{in,j}+a_{jn,i}-a_{ij,n})y^j=S_{,i}l_n+(\theta_{,i}+p_i)Sm_n.$$

Thus we have

$$a^n{}_{ij}y^j=S_{,i}l^n+(\theta_{,i}+p_i)Sm^n \tag{A.9}$$

which is the representation (2.25).

From (A.8) we may observe that

$$\theta(x,b(x))=0.$$

Appendix B: Finsleroid-regular space with connection

We start with the notion of a 2-dimensional Riemannian space $\mathcal{R}_2 = (M, \mathcal{S})$ to specify an attractive 2-dimensional Finsler space over $\mathcal{R}_2$. We shall assume that the background two-dimensional manifold M admits introducing two linearly independent covariant vector fields, to be presented by the non–vanishing 1-forms $\widetilde{b} = \widetilde{b}(x,y)$ and $n = n(x,y)$. With respect to natural local coordinates $\{x^n\}$ in the manifold M the expansions

$$\widetilde{b} = \widetilde{b}_i(x)y^i, \qquad n = n_i(x)y^i, \qquad S = \sqrt{a_{ij}(x)y^iy^j}, \tag{B.1}$$

represent the 1-forms and the Riemannian metric $\mathcal{S}$, with a_{ij} standing for the covariant components of the Riemannian metric tensor of the space $\mathcal{R}_2$. The contravariant components a^{ij} are defined by means of the reciprocity $a^{in}a_{nj} = \delta^i{}_j$. The covariant index of the vectors $\widetilde{b}_i$ and n_i will be raised by means of the Riemannian rule $\widetilde{b}^i = a^{ij}\widetilde{b}_j$, $n^i = a^{ij}n_j$, which inverse reads $\widetilde{b}_i = a_{ij}\widetilde{b}^j$, $n_i = a_{ij}n^j$. Also, we assume that the vectors introduced are orthonormal relative to the Riemannian metric $\mathcal{S}$, that is,

$$a_{mn}\widetilde{b}^m\widetilde{b}^n = 1, \qquad a_{mn}n^mn^n = 1, \qquad a_{mn}\widetilde{b}^mn^n = 0. \tag{B.2}$$

Let a positive scalar $c = c(x)$ be given which is ranged as follows:

$$0 < c < 1. \tag{B.3}$$

We are entitled to construct the 1-form

$$b = b_i(x)y^i = c\widetilde{b}, \tag{B.4}$$

so that

$$\widetilde{b}_i = \frac{1}{c}b_i, \qquad \widetilde{b}^i = \frac{1}{c}b^i, \qquad c = ||b||_{\text{Riemannian}} \equiv \sqrt{a^{mn}b_mb_n}, \tag{B.5}$$

where $b^i = a^{ij}b_j$.

Since $c < 1$, we get

$$S^2 - b^2 > 0 \quad \text{whenever} \quad y \neq 0 \tag{B.6}$$

and may conveniently use the variable

$$q := \sqrt{S^2 - b^2} \tag{B.7}$$

which *does not vanish anywhere on* $T_xM\backslash 0$. Obviously, the inequality

$$q^2 \geq \frac{1-c^2}{c^2}\,b^2 \tag{B.8}$$

is valid.

We have

$$q = \sqrt{\frac{1-c^2}{c^2}b^2 + n^2}, \tag{B.9}$$

so that

$$q\frac{\partial q}{\partial y^m} = \frac{1-c^2}{c^2}bb_m + nn_m \tag{B.10}$$

and

$$\frac{\partial \dfrac{b}{q}}{\partial y^m} = \frac{1}{q} b_m - \frac{b}{q^3}\left(\frac{1-c^2}{c^2} b b_m + n n_m\right),$$

or

$$\frac{\partial \dfrac{b}{q}}{\partial y^m} = -\frac{n}{q^3}(b n_m - n b_m). \tag{B.11}$$

The space described below involves the characteristic parameter

$$g = g(x) \in (-2, 2). \tag{B.12}$$

It is convenient to introduce the quantities

$$h = \sqrt{1 - \frac{g^2}{4}}, \qquad G = \frac{g}{h}. \tag{B.13}$$

The functions $B = B(x, y)$ and $B_1 = B_1(x, y)$ are specified as follows:

$$B = b^2 + gbq + q^2, \qquad B_1 = \frac{1}{c^2}(b + gc^2 q), \qquad B - bB_1 = n^2. \tag{B.14}$$

The function B is *positively definite*, for

$$b^2 + gqb + q^2 = \frac{1}{2}\left[(b + g_+ q)^2 + (b + g_- q)^2\right],$$

where $g_+ = (1/2)g + h$ and $g_- = (1/2)g - h$. In the limit $g \to 0$, the function B degenerates to the quadratic form of the input Riemannian metric tensor:

$$B\big|_{g=0} = b^2 + q^2 \equiv S^2. \tag{B.15}$$

Also,

$$n\big|_{y^n=b^n} = 0, \quad b\big|_{y^n=b^n} = c^2, \quad q\big|_{y^n=b^n} = c\sqrt{1-c^2}, \quad S^2\big|_{y^n=b^n} = c^2, \quad \eta B\big|_{y^n=b^n} = c^2, \tag{B.16}$$

where

$$\eta = \frac{1}{1 + gc\sqrt{1-c^2}}. \tag{B.17}$$

It can readily be verified that on the definition range $g \in (-2, 2)$ of the parameter g we have $\eta > 0$.

The basic metric function K comes from the following definition.

Key Definition. The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)}\, J(x, y), \qquad J(x, y) = \mathrm{e}^{-\frac{1}{2}G(x) f(x,y)}, \tag{B.18}$$

where

$$f = -\arctan\frac{G}{2} + \arctan\frac{L}{hb}, \qquad \text{if} \quad b \geq 0, \tag{B.19}$$

and

$$f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \qquad \text{if} \quad b \le 0, \tag{B.20}$$

with

$$L = q + \frac{g}{2}b, \tag{B.21}$$

is called the *Finsleroid-regular metric function.*

This metric function K is not absolute homogeneous.

The function L obeys the identity

$$L^2 + h^2b^2 = B. \tag{B.22}$$

Definition. The arisen Finsler space

$$\mathcal{FR}^{PD}_{g;c} := \{\mathcal{R}_2;\ b_i(x);\ g(x);\ K(x,y)\} \tag{B.23}$$

is called the *Finsleroid-regular space.*

The upperscripts PD mean that the space is positive-definite.

Definition. Within any tangent space T_xM, the metric function $K(x,y)$ produces the $\mathcal{FR}^{PD}_{g;c}$*-circle*

$$\mathcal{FR}^{PD}_{g;c\{x\}} := \{y \in \mathcal{FR}^{PD}_{g;c\{x\}} : y \in T_xM, K(x,y) \le 1\}. \tag{B.24}$$

Definition. The $\mathcal{FR}^{PD}_{g;c}$*-indicatrix* $I\mathcal{R}^{PD}_{g;c\{x\}} \subset T_xM$ is the boundary of the $\mathcal{FR}^{PD}_{g;c}$-circle, that is,

$$I\mathcal{R}^{PD}_{g;c\{x\}} := \{y \in I\mathcal{R}^{PD}_{g;c\{x\}} : y \in T_xM, K(x,y) = 1\}. \tag{B.25}$$

Definition. The scalar $g(x)$ is called the *Finsleroid charge.* The 1-form $b = b_i(x)y^i$ is called the $\mathcal{FR}^{PD}_{g;c}$*-axis one-form.*

The indicatrix (B.25) is obviously symmetric under reflections with respect to the direction assigned by the vector b_i.

The metric function K given by (B.18)-(B.21) is regular of the class C^∞ regarding the y-dependence. Formally, this high regularity comes from the circumstance that $q > 0$ and the derivatives $\partial q/\partial y^m$ and $\partial(b/q)/\partial y^m$ (see (B.10) and (B.11)) are of this class C^∞. The made assumption $c \in (0,1)$ is essential, for at $c = 1$ the quantity q may vanish (at $n = 0$, that is, on the axis of the indicatrix).

We shall meet the function

$$\nu := q + (1 - c^2)gb \tag{B.26}$$

for which

$$\nu > 0 \quad \text{when} \quad |g| < 2. \tag{B.27}$$

Indeed, if $gb > 0$, then the right-hand part of (B.26) is positive. When $gb < 0$, we may note that at any fixed c and b the minimal value of q equals $\sqrt{1-c^2}|b|/c$ (see (B.8)), arriving again at (B.27).

The identities

$$\frac{c^2S^2-b^2}{q\nu} = 1-(1-c^2)\frac{B}{q\nu}, \qquad gb(c^2S^2-b^2) = qB-\nu S^2 \tag{B.28}$$

are of great help to take into account when performing evaluations of key objects entailed.

We straightforwardly obtain the representations

$$y_i \ := \frac{1}{2}\frac{\partial K^2}{\partial y^i} = (u_i+gqb_i)\frac{K^2}{B}, \qquad u_i = a_{ij}y^j, \tag{B.29}$$

and

$$g_{ij} \ := \frac{\partial y_i}{\partial y^j} = \left[a_{ij}+\frac{g}{B}\left(\left(gq^2-\frac{bS^2}{q}\right)b_ib_j-\frac{b}{q}u_iu_j+\frac{S^2}{q}(b_iu_j+b_ju_i)\right)\right]\frac{K^2}{B}, \tag{B.30}$$

together with the reciprocal components

$$g^{ij} = \left[a^{ij}+\frac{g}{\nu}(bb^ib^j-b^iy^j-b^jy^i)+\frac{g}{B\nu}(b+gc^2q)y^iy^j\right]\frac{B}{K^2}. \tag{B.31}$$

The associated Riemannian metric tensor a_{ij} has the meaning

$$a_{ij} = g_{ij}\big|_{g=0}. \tag{B.32}$$

The determinant of the Finsler metric tensor presented by (B.30) is everywhere positive:

$$\det(g_{ij}) = \frac{\nu}{q}\left(\frac{K^2}{B}\right)^2\det(a_{ij}) > 0. \tag{B.33}$$

Using the function X given by

$$\frac{1}{X} = 2+\frac{(1-c^2)B}{q\nu} \equiv 3-\frac{c^2n^2}{q\nu}, \tag{B.34}$$

we obtain

$$A_i \ := K\frac{\partial \ln\left(\sqrt{\det(g_{mn})}\right)}{\partial y^i} = \frac{Kg}{2qB}\frac{1}{X}(S^2b_i-bu_i) \equiv \frac{Kg}{2qB}\frac{n}{X}(nb_i-bn_i) \tag{B.35}$$

and

$$A^i \ := g^{ij}A_j = \frac{g}{2XK\nu}\left[Bb^i-(b+gqc^2)y^i\right] \equiv \frac{g}{2XK\nu}(Bb^i-c^2B_1y^i), \tag{B.36}$$

which entails the contraction

$$A^iA_i = \frac{g^2}{4X^2}\left(3 - \frac{1}{X}\right) \equiv \frac{g^2}{4X^2}\frac{c^2n^2}{q\nu}. \tag{B.37}$$

With the function

$$T = \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{K^2}{B} \equiv \frac{1}{c}\sqrt{\frac{\det(g_{ij})}{\det(a_{mn})}}, \tag{B.38}$$

we introduce the vector

$$m_n = \frac{1}{K}T(bn_n - nb_n) \tag{B.39}$$

which coincides with the vector (2.1). The contravariant components $m^n = g^{nk}m_k$ are found with the help of (B.31) to read

$$m^n = c\sqrt{\frac{q}{\nu}}\left(-\frac{1}{c^2}nb^n + \left(\frac{1}{c^2}b + gq\right)n^n\right)\frac{1}{K},$$

or

$$m^n = c\sqrt{\frac{q}{\nu}}\left(\frac{1}{c^2}(bn^n - nb^n) + gqn^n\right)\frac{1}{K}. \tag{B.40}$$

We can write also

$$bm^n = c\sqrt{\frac{q}{\nu}}(Bn^n - ny^n)\frac{1}{K}.$$

The unit norm equality

$$m^nm_n = 1$$

is valid.

Comparing (B.35) and (B.37) with (B.39) shows that

$$\frac{A_k}{\sqrt{A_hA^h}} = m_k\gamma \quad \text{whenever } g \neq 0 \text{ and } n \neq 0,$$

with

$$\gamma = -\operatorname{sign}(g)\operatorname{sign}(n),$$

where the designation ‘ sign ‘ stands for the function: $\operatorname{sign}(x) = 1$, if $x > 0$, and $\operatorname{sign}(x) = -1$, if $x < 0$.

From (B.35) and (B.39) the representation

$$A_i = Im_i \tag{B.41}$$

ensues, where

$$I = -gc\frac{1}{2X}\frac{n}{\sqrt{q\nu}}. \tag{B.42}$$

We shall use the angle function θ defined by the equation

$$K\frac{\partial\theta}{\partial y^n} = m_n, \tag{B.43}$$

where the components m_n placed in the right-hand part are taken from (B.39).

The I given by (B.42) and the components $A_i = A_i(x, y)$ reveal the vanishing at the values $y^n = \pm b^n(x)$:

$$I(x, b(x)) = I(x, -b(x)) = 0, \qquad A_i(x, b(x)) = A_i(x, -b(x)) = 0,$$

that is, on the indicatrix axis (and nowhere else), because they involve the factor n.

At the same time, on all the slit tangent bundle $T_xM\backslash 0$ the vector m_k introduced in (B.39) is C^∞-regular regarding the dependence on tangent vectors.

Since

$$K\big|_{y^n=b^n} = c\frac{1}{\sqrt{\eta}}\,\mathrm{e}^{-\frac{1}{2}Gj}$$

with

$$j = -\arctan\frac{G}{2} + \arctan\frac{\sqrt{1-c^2}+\dfrac{1}{2}gc}{hc},$$

where we have used (B.18) and (B.19), it may be convenient to rescale the metric function K as follows:

$$K \to \breve{K} = \sqrt{\eta}\,\mathrm{e}^{\frac{1}{2}Gj}\,K. \tag{B.44}$$

We obtain the function

$$\breve{K}(x,y) = \sqrt{\eta B(x,y)}\,\breve{J}(x,y), \qquad \breve{J}(x,y) = \mathrm{e}^{-\frac{1}{2}G(x)\breve{f}(x,y)}, \tag{B.45}$$

where

$$\breve{f} = \arctan\frac{L}{hb} - \arctan\frac{\sqrt{1-c^2}+\dfrac{1}{2}gc}{hc}, \qquad \text{if}\quad b\ge 0, \tag{B.46}$$

and

$$\breve{f} = \pi + \arctan\frac{L}{hb} - \arctan\frac{\sqrt{1-c^2}+\dfrac{1}{2}gc}{hc}, \qquad \text{if}\quad b\le 0, \tag{B.47}$$

which possesses the property

$$\breve{K}\big|_{y^n=b^n} = c. \tag{B.48}$$

In terms of the metric function normalized in this way the norm of the input vector b^i is the same with respect to the considered Finsler space as well as the underlined Riemannian space $\mathcal{R}_2$, that is,

$$\breve{K}(x,b(x)) = \sqrt{a_{ij}(x)b^i(x)b^j(x)} = c(x). \tag{B.49}$$

From (B.30) we get

$$g_{ij}\big|_{y^n=b^n} = \left[a_{ij} + g\eta\Big(gc^2(1-c^2) - \frac{c^4}{c\sqrt{1-c^2}} - \frac{c^2}{c\sqrt{1-c^2}} + 2\frac{c^2}{c\sqrt{1-c^2}}\Big)\widetilde{b}_i\widetilde{b}_j\right]\frac{\eta}{c^2}K^2\big|_{y^n=b^n}$$

$$= \left[a_{ij} + g\eta\Big(gc^2(1-c^2) + c\sqrt{1-c^2}\Big)\widetilde{b}_i\widetilde{b}_j\right]\frac{\eta}{c^2}K^2\big|_{y^n=b^n},$$

or

$$g_{ij}\big|_{y^n=b^n} = \Big(a_{ij} + gc\sqrt{1-c^2}\,\widetilde{b}_i\widetilde{b}_j\Big)\,\frac{\eta}{c^2}K^2\big|_{y^n=b^n}.$$

Introducing the osculating tensor

$$\breve{a}_{ij} \;:=\; g_{ij}\big|_{y^n=b^n} \tag{B.50}$$

leads therefore to

$$a_{ij} = \mathrm{e}^{Gj}\,\breve{a}_{ij} - gc\sqrt{1-c^2}\,\widetilde{b}_i\widetilde{b}_j. \tag{B.51}$$

We may make here the expansion

$$\breve{a}_{ij} = \breve{b}_i \breve{b}_j + \breve{n}_i \breve{n}_j,$$

obtaining

$$\breve{b}_i = \frac{1}{\sqrt{\eta}} \, \mathrm{e}^{-\frac{1}{2}Gj} \, \widetilde{b}_i, \qquad \breve{n}_i = \mathrm{e}^{-\frac{1}{2}Gj} \, n_i. \tag{B.52}$$

In terms of the new variables $\{\breve{n} = \breve{n}_i y^i,\ \breve{b} = \breve{b}_i y^i\}$ we can replace the initial quantities $q = \sqrt{S^2 - b^2}$ and $S = \sqrt{a_{ij} y^i y^j}$ by the quantities $\breve{q} = \sqrt{\breve{S}^2 - \breve{b}^2}$ and $\breve{S} = \sqrt{\breve{a}_{ij} y^i y^j}$ which are adaptable to the treatment of the osculating tensor $\breve{a}_{ij}$ to be the metric tensor of the background Riemannian space. In this vein the derivative values

$$\frac{\partial \breve{\theta}}{\partial y^i} \left(x, \breve{b}(x) \right) = \breve{n}_i(x), \qquad \frac{\partial \breve{\theta}}{\partial y^i} \left(x, -\breve{b}(x) \right) = -\breve{n}_i(x) \tag{B.53}$$

are obtained (cf. (4.18)).

Now, differentiating Km^k taken from (B.40) yields

$$\frac{\partial K m^k}{\partial y^m} = -Km^k \frac{q}{2\nu}(1-c^2) g \frac{\partial \frac{b}{q}}{\partial y^m} + c\sqrt{\frac{q}{\nu}} \left(\frac{1}{c^2}(b_m n^k - n_m b^k) + g \frac{\partial q}{\partial y^m} n^k \right),$$

so that we can write

$$\frac{\partial K m^k}{\partial y^m} - l_m m^k + l^k m_m = Km^k \frac{q}{2\nu}(1-c^2) g \frac{n}{q^3} \Upsilon_m$$

$$+c\sqrt{\frac{q}{\nu}} \left(\frac{1}{c^2} \Upsilon^k_m + g \frac{1}{q} \left(\frac{1-c^2}{c^2} b b_m + n n_m \right) n^k \right) - l_m m^k + l^k m_m.$$

We use the notation

$$\Upsilon_m = b n_m - n b_m, \qquad \Upsilon^k = b n^k - n b^k, \qquad \Upsilon^k_m = b_m n^k - n_m b^k.$$

We straightforwardly have

$$-l_m m^k + l^k m_m = \frac{1}{B} c \sqrt{\frac{q}{\nu}} \, p^k_m$$

with

$$p^k_m = \frac{\nu}{q} \, y^k \frac{1}{c^2} \Upsilon_m - \left(\frac{1}{c^2} \Upsilon^k + g q n^k \right) (u_m + g q b_m)$$

$$= b b^k \frac{1}{c^4}(b n_m - n b_m) + n n^k \frac{1}{c^2}(b n_m - n b_m) + \frac{1-c^2}{c^4} g \frac{b}{q} b b^k \Upsilon_m + (1-c^2) g \frac{b}{q} n n^k \frac{1}{c^2} \Upsilon_m$$

$$-\left(\frac{1}{c^2}(bn^k-nb^k)+gqn^k\right)\frac{b}{c^2}b_m-\left(\frac{1}{c^2}(bn^k-nb^k)+gqn^k\right)nn_m-g\left(\frac{1}{c^2}\Upsilon^k+gqn^k\right)qb_m.$$

Canceling similar terms leaves us with merely

$$p^k_m=-\frac{1}{c^2}B\Upsilon^k_m-n^2g\frac{1}{q}b^k\frac{1}{c^2}\Upsilon_m+(1-c^2)g\frac{b}{q}nn^k\frac{1}{c^2}\Upsilon_m-gqnn^kn_m-gqn^k\frac{1}{c^2}bb_m-g^2q^2n^kb_m. \tag{B.54}$$

With this formula we find that

$$\frac{\partial Km^k}{\partial y^m}-l_mm^k+l^km_m=Km^k\frac{q}{2\nu}(1-c^2)g\frac{n}{q^3}\Upsilon_m+g\frac{1}{B}\sqrt{\frac{q}{\nu}}\frac{c}{q}\left(-b^k\frac{n^2}{c^2}+(1-c^2)bnn^k\frac{1}{c^2}\right)\Upsilon_m$$

$$+g\frac{gc}{Bq}\sqrt{\frac{q}{\nu}}\left[-\frac{q^2}{c^2}bb_m-gq^3b_m+(b^2+gbq)\frac{1-c^2}{c^2}bb_m+q^2\frac{1-c^2}{c^2}bb_m+(b^2+gbq)nn_m\right]n^k$$

$$=Km^k\frac{q}{2\nu}(1-c^2)g\frac{n}{q^3}\Upsilon_m+g\frac{1}{B}c\sqrt{\frac{q}{\nu}}\frac{1}{q}\left(-n^2b^k\frac{1}{c^2}+(1-c^2)bnn^k\frac{1}{c^2}\right)\Upsilon_m$$

$$+g\frac{gc}{Bq}\sqrt{\frac{q}{\nu}}(b+gq)\left[b\frac{1-c^2}{c^2}bb_m-q^2b_m+bnn_m\right]n^k,$$

or

$$\frac{\partial Km^k}{\partial y^m}-l_mm^k+l^km_m=Km^k\frac{q}{2\nu}(1-c^2)g\frac{n}{q^3}\Upsilon_m$$

$$+g\frac{1}{B}c\sqrt{\frac{q}{\nu}}\frac{1}{q}n\left(-nb^k\frac{1}{c^2}+(1-c^2)bn^k\frac{1}{c^2}\right)\Upsilon_m+g\frac{gc}{Bq}\sqrt{\frac{q}{\nu}}(b+gq)n\Upsilon_mn^k,$$

so that

$$\frac{\partial Km^k}{\partial y^m}-l_mm^k+l^km_m=Km^k\frac{q}{2\nu}(1-c^2)g\frac{n}{q^3}(bn_m-nb_m)+g\frac{1}{B}\frac{1}{q}nKm^k(bn_m-nb_m).$$

Using here the equality

$$bn_m-nb_m=cB\frac{1}{K}\sqrt{\frac{q}{\nu}}m_m$$

leads to

$$\frac{\partial Km^k}{\partial y^m}-l_mm^k+l^km_m=gm^km_m\left[\frac{q}{2\nu}(1-c^2)\frac{n}{q^3}B+\frac{1}{q}n\right]c\sqrt{\frac{q}{\nu}}.$$

The eventual result reads

$$\frac{\partial K m^k}{\partial y^m} = l_m m^k - l^k m_m + gcm^k m_m \frac{1}{2X}\frac{n}{\sqrt{q\nu}}. \tag{B.55}$$

Next, with (B.38) we find that

$$\frac{\partial \frac{1}{K}T}{\partial y^i} = l_i \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{1}{B} - \frac{1}{2c}\sqrt{\frac{q}{\nu}}(1-c^2)g\frac{n}{q^3}\Upsilon_i\frac{K}{B}$$

$$-\frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{K}{B^2}\left[2bb_i + 2\left(\frac{1-c^2}{c^2}bb_i + nn_i\right) + gb_i q + \frac{1}{q}g\left(\frac{1-c^2}{c^2}b^2 b_i + bnn_i\right)\right],$$

or

$$\frac{\partial \frac{1}{K}T}{\partial y^i} = -l_i \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{1}{B} - \frac{1}{2c}\sqrt{\frac{q}{\nu}}(1-c^2)g\frac{n}{q^3}\Upsilon_i\frac{K}{B} - \frac{1}{c}q\nu\frac{1}{\sqrt{\nu q}}\frac{K}{B^2}\frac{1}{q^2}ng\Upsilon_i. \tag{B.56}$$

Therefore, from (B.39) it follows that

$$\frac{\partial m_m}{\partial y^i} = \left[-l_i\frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{1}{B} - \frac{1}{c}\left[q\nu + \frac{1}{2}(1-c^2)B\right]\frac{1}{\sqrt{\nu q}}\frac{K}{B^2}\frac{1}{q^2}ng\Upsilon_i\right]\Upsilon_m + \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{K}{B}(b_i n_m - b_m n_i)$$

$$= -\frac{1}{c}\left[q\nu + \frac{1}{2}(1-c^2)B\right]\frac{1}{\sqrt{\nu q}}\frac{K}{B^2}\frac{1}{q^2}ng(bn_i - nb_i)(bn_m - nb_m) + \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{K}{B^2}Y_{im},$$

where

$$Y_{im} = B(b_i n_m - n_i b_m) - \left(\frac{1}{c^2}bb_i + nn_i + gqb_i\right)(bn_m - nb_m) = -\frac{B}{K}l_m(bn_i - nb_i).$$

Using also $bn_m - nb_m = c(B/K)\sqrt{q/\nu}\, m_m$, we arrive at the result

$$K\frac{\partial m_m}{\partial y^i} = -gcm_m m_i \frac{1}{2X}\frac{n}{\sqrt{q\nu}} - l_m m_i. \tag{B.57}$$

Differentiating the tensor $h_{ij} = g_{ij} - l_i l_j$ leads to

$$K\frac{\partial h_{ij}}{\partial y^k} = 2A_{ijk} - l_i h_{jk} - l_j h_{ik},$$

where $A_{ijk} = KC_{ijk}$. Applying here (B.57) yields the representation

$$A_{ijk} = -gc\frac{1}{2X}\frac{n}{\sqrt{q\nu}}m_i m_j m_k.$$

We may write

$$A_{ijk} = \frac{1}{A_h A^h} A_i A_j A_k \quad \text{whenever } g \neq 0 \ \text{ and } n \neq 0.$$

We shall use the derivative

$$\partial^*_n = g_n \frac{\partial}{\partial g} \quad \text{with} \quad g_n = \frac{\partial g}{\partial x^n}. \tag{B.58}$$

When the function K given by (B.18) is differentiated with respect to x^n, we obtain

$$\frac{\partial K}{\partial x^n} = \partial^*_n K + \frac{K}{B} g q y^j \nabla_n b_j + a^k{}_{nj} y^j l_k. \tag{B.59}$$

Let us verify this formula. With (B.18)-(B.21) we find

$$\frac{\partial K^2}{\partial x^n} = \partial^*_n K^2 + \left[\partial_n S^2 + g(b\partial_n q + q\partial_n b) - g(b\partial_n q - q\partial_n b)\right] J^2 = \partial^*_n K^2 + (\partial_n S^2 + 2gq\partial_n b) J^2$$

(∂_n means $\partial/\partial x^n$), so that

$$\frac{\partial K}{\partial x^n} = \partial^*_n K + \frac{1}{2}\frac{K}{B} y^i y^j \partial_n a_{ij} + \frac{K}{B} g q \partial_n b.$$

We obtain directly that

$$\frac{1}{2}\frac{K}{B} y^i y^j \partial_n a_{ij} - a^k{}_{nj} y^j l_k = \frac{1}{2}\frac{K}{B} y^j \left[y^i \partial_n a_{ij} - (y^k + gqb^k)(\partial_n a_{kj} + \partial_j a_{kn} - \partial_k a_{nj})\right],$$

or

$$\frac{1}{2}\frac{K}{B} y^i y^j \partial_n a_{ij} - a^k{}_{nj} y^j l_k = -\frac{K}{B} y^j g q b_k a^k{}_{nj}.$$

The formula (B.59) is valid.

The equality $\partial K^2/\partial g = \bar{M} K^2$ holds with

$$\bar{M} = -\frac{1}{h^3} f + \frac{1}{2}\frac{G}{hB} q^2 + \frac{1}{h^2 B} bq. \tag{B.60}$$

In obtaining this formula we have used the derivatives

$$\frac{\partial h}{\partial g} = -\frac{1}{4}G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}, \quad \frac{\partial\left(\dfrac{G}{h}\right)}{\partial g} = \frac{1}{h^4}\left(1 + \frac{g^2}{4}\right), \quad \frac{\partial f}{\partial g} = -\frac{1}{2h} + \frac{b}{B}\left(\frac{1}{4}Gq + \frac{1}{2h}b\right).$$

Therefore,

$$\partial^*_n K = \bar{M} K^2 \frac{\partial g}{\partial x^n}. \tag{B.61}$$

It follows that

$$\frac{\partial \bar{M}}{\partial y^h} = \frac{2b^4}{B^2} \frac{\partial \dfrac{b}{q}}{\partial y^m} = \frac{4q^2 X}{gBK} A_h$$

and

$$\partial^*_n l_m = \frac{\partial(\partial^*_n K)}{\partial y^m}.$$

All the right-hand parts in the equalities (B.59)-(B.61) are regular of class C^∞ globally regarding the y-dependence.

To elucidate the global properties of the angle function $\theta=\theta(x,y)$ in tangent spaces, we note that at any point $x\in M$ the space $T_xM\backslash 0$ can conveniently be covered by the atlas

$$\mathcal{A}=\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3\cup\mathcal{C}_4 \tag{B.62}$$

of the four charts

$$\mathcal{C}_1=\{y\in\mathcal{C}_1: b>0\},\ \mathcal{C}_2=\{y\in\mathcal{C}_2: n>0\},\ \mathcal{C}_3=\{y\in\mathcal{C}_3: n<0\},\ \mathcal{C}_4=\{y\in\mathcal{C}_4: b<0\}. \tag{B.63}$$

The north pole and the south pole of the indicatrix (B.25) belong to the charts $\mathcal{C}_1$ and $\mathcal{C}_4$, respectively. In the region $\mathcal{C}_1\cup\mathcal{C}_4$ the variable

$$\widetilde{w}=\frac{n}{b} \tag{B.64}$$

can naturally be used to introduce the function $\widetilde{\theta}(x,\widetilde{w})$ defined by $\theta=\widetilde{\theta}$. We have

$$\frac{\partial\widetilde{w}}{\partial y^n}=\frac{bn_n-nb_n}{b^2}. \tag{B.65}$$

Comparing this with (B.39) and (B.43), we conclude that if $y\in\mathcal{C}_1\cup\mathcal{C}_4$, then

$$\frac{\partial\widetilde{\theta}}{\partial\widetilde{w}}=\frac{1}{c}\frac{b^2}{B}\sqrt{\frac{\nu}{q}}. \tag{B.66}$$

In alternative regions the variable

$$t=-\frac{b}{q},\quad \text{if}\quad y\in\mathcal{C}_2;\qquad t=\frac{b}{q},\quad \text{if}\quad y\in\mathcal{C}_3, \tag{B.67}$$

is well adaptable. Since

$$\frac{\partial t}{\partial y^m}=\frac{|n|}{q^3}(bn_m-nb_m) \tag{B.68}$$

(see (B.11)), we obtain that if $y\in\mathcal{C}_2\cup\mathcal{C}_3$, then

$$\frac{\partial\widehat{\theta}}{\partial t}=\frac{1}{c}\frac{q^3}{|n|B}\sqrt{\frac{\nu}{q}}, \tag{B.69}$$

when using the function $\widehat{\theta}(x,t)$ defined by $\theta=\widehat{\theta}$. The positivity

$$\frac{\partial\widetilde{\theta}}{\partial\widetilde{w}}>0,\quad \frac{\partial\widehat{\theta}}{\partial t}>0 \tag{B.70}$$

holds.

It can readily be seen that in intersections of the charts introduced the variables $\widetilde{w}$ and t are expressible one through another in the C^∞-smooth way, so that the atlas $\mathcal{A}$ introduced in (B.62) is smooth of the class C^∞. The right-hand parts of the angle derivatives (B.66) and (B.69) are also of this class with respect to y. Noting also the

positivity (B.70), we are entitled to conclude that when point moves along the indicatrix from the north pole in the right direction the value θ of the point increases monotonically

$$0 \le \theta < \theta^{\max}, \tag{B.71}$$

where

$$\theta^{\max} = 2(\theta^I + \theta^{II}) \tag{B.72}$$

with

$$\theta^I = \int_0^\infty \frac{1}{c}\frac{b^2}{B}\sqrt{\frac{\nu}{q}}d\widetilde{w}, \qquad \theta^{II} = \int_{-\infty}^0 \frac{1}{c}\frac{b^2}{B}\sqrt{\frac{\nu}{q}}d\widetilde{w}. \tag{B.73}$$

Reminding the expressions of the functions B and ν (see (B.14) and (B.26)), we can write these integrals explicitly as follows:

$$\theta^I = \frac{1}{c}\int_0^\infty \frac{\sqrt{1+(1-c^2)g\dfrac{1}{w}}}{1+gw+w^2}d\widetilde{w}, \qquad \theta^{II} = \frac{1}{c}\int_0^\infty \frac{\sqrt{1-(1-c^2)g\dfrac{1}{w}}}{1-gw+w^2}d\widetilde{w}, \tag{B.74}$$

where

$$w = \sqrt{\widetilde{w}^2 + \frac{1-c^2}{c^2}} \equiv \frac{q}{|b|}. \tag{B.75}$$

The value $\theta = 0$ corresponds to the north pole of indicatrix, and the value $\theta = \theta^I + \theta^{II}$ to the south pole. The indicatrix intersects the direction with $b = 0, n > 0$ at $\theta = \theta^I$, so that the angle measure of the upper chart $\mathcal{C}_1$ equals $2\theta^I$, respectively $2\theta^{II}$ for the chart $\mathcal{C}_4$. The value $\theta^{\max} = 2(\theta^I + \theta^{II})$ is the total length of the indicatrix. The vertical straight angle going from the north to the south costs $\theta^I + \theta^{II}$.

In general, $\theta^{\max} = \theta^{\max}(x)$.

In the Riemannian limit we obtain $\theta^I = \theta^{II} = \pi/2$ (put $g = 0$ and $c = 1$ in (B.74)).

In the region $\mathcal{C}_1$ we can interpret the $\widetilde{\theta}$ to be of the functional dependence

$$\widetilde{\theta} = \widetilde{\Theta}\big(g(x), c(x), \widetilde{w}\big), \tag{B.76}$$

obtaining from (B.66)

$$\widetilde{\Theta} = \frac{1}{c}\int \frac{\sqrt{1+(1-c^2)g\dfrac{1}{w}}}{1+gw+w^2}d\widetilde{w}. \tag{B.77}$$

The integration constant should be specified by means of the condition

$$\widetilde{\theta}_{\big|\widetilde{w}=0} = 0 \tag{B.78}$$

to be in agreement with $\theta(x, b(x)) = 0$. Differentiating the equality $\theta(x, y) = \widetilde{\Theta}\big(g(x), c(x), \widetilde{w}\big)$ with respect to x^n yields

$$\frac{\partial\theta}{\partial x^n} = \frac{\partial\widetilde{\Theta}}{\partial g}\frac{\partial g}{\partial x^n} + \frac{\partial\widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} + \frac{\partial\widetilde{\Theta}}{\partial \widetilde{w}}\frac{\partial \widetilde{w}}{\partial x^n}. \tag{B.79}$$

From (B.77) it follows that

$$\frac{\partial\widetilde{\Theta}}{\partial g} = \frac{1}{c}\int\left(-\frac{w\sqrt{1+(1-c^2)g\dfrac{1}{w}}}{(1+gw+w^2)^2} + \frac{(1-c^2)}{2w(1+gw+w^2)}\frac{1}{\sqrt{1+(1-c^2)g\dfrac{1}{w}}}\right)d\widetilde{w} \tag{B.80}$$

and

$$\frac{\partial \widetilde{\Theta}}{\partial c} = -\frac{1}{c}\widetilde{\Theta} - g\int \frac{1}{w}\frac{1}{(1+gw+w^2)\sqrt{1+(1-c^2)g\dfrac{1}{w}}}d\widetilde{w}. \tag{B.81}$$

Using (B.76), (B.66), and (B.39), we can write

$$\frac{\partial \theta}{\partial x^n} = \frac{\partial \widetilde{\Theta}}{\partial g}\frac{\partial g}{\partial x^n} + \frac{\partial \widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} + \frac{1}{K^2}Tb^2\left(\frac{\partial \widetilde{w}}{\partial x^n} - a^k{}_{nj}y^j(bn_k - nb_k)\frac{1}{b^2}\right) + a^k{}_{nj}y^j\frac{\partial \theta}{\partial y^k}. \tag{B.82}$$

Taking into account equality

$$\frac{\partial \widetilde{w}}{\partial x^n} = y^k\frac{1}{b^2}(b\nabla_n n_k - n\nabla_n b_k) + \frac{1}{b^2}(bn_k - nb_k)a^k{}_{nj}y^j,$$

we obtain

$$\frac{\partial \widetilde{w}}{\partial x^n} - a^k{}_{nj}y^j(bn_k - nb_k)\frac{1}{b^2} = y^k\frac{1}{b^2}(b\nabla_n n_k - n\nabla_n b_k),$$

which yields

$$b^2\left(\frac{\partial \widetilde{w}}{\partial x^n} - a^k{}_{nj}y^j(bn_k - nb_k)\frac{1}{b^2}\right) = -S^2cn^h\nabla_n\widetilde{b}_h - nb\frac{1}{c}\frac{\partial c}{\partial x^n}.$$

Inserting this result in (B.82), we are coming to

$$\frac{\partial \theta}{\partial x^n} = \frac{\partial \widetilde{\Theta}}{\partial g}\frac{\partial g}{\partial x^n} + \frac{\partial \widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} - \frac{1}{K^2}T\left(S^2cn^h\nabla_n\widetilde{b}_h + bn\frac{1}{c}\frac{\partial c}{\partial x^n}\right) + a^k{}_{nj}y^j\frac{\partial \theta}{\partial y^k}. \tag{B.83}$$

Here, all the terms are smooth of class C^∞ regarding the y-dependence. Similar representations for the derivative $\partial\theta/\partial x^n$ can be obtained in the regions $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$.

Henceforth, we assume

$$c = const. \tag{B.84}$$

Let us propose the expansion

$$N^k_n = \left[\left(\frac{1}{c^2}b - Uc^2B_1\right)n^k + n\left(U - \frac{1}{c^2}\right)b^k\right]p_n - l^k\partial^*_n K - Km^k\partial^*_n\theta - a^k{}_{nj}y^j, \tag{B.85}$$

where $B_1 = (1/c^2)b + gq$ (in agreement with (B.14)) and

$$p_n = cn^h\nabla_n\widetilde{b}_h \equiv cn^h\left(\frac{\partial \widetilde{b}_h}{\partial x^n} - a^k{}_{nh}\widetilde{b}_k\right). \tag{B.86}$$

By contracting the coefficients N^k_n written in (B.85) we obtain

$$N^k_n y_k = -K\partial^*_n K - gnqp_n\frac{K^2}{B} - a^k{}_{nj}y^jl_k. \tag{B.87}$$

On the other hand,

$$\frac{\partial K}{\partial x^n} = \partial^*_n K + \frac{K}{B}gnqp_n + a^k{}_{nj}y^jl_k \tag{B.88}$$

(see (B.59)). Comparing this equality with (B.87) just yields

$$d_n K = 0. \tag{B.89}$$

If we take in (B.85) the function U to be

$$U = C_1 \frac{1}{c}\sqrt{\frac{q}{\nu}}, \qquad C_1 = -\frac{1}{c}, \tag{B.90}$$

we obtain

$$\frac{\partial \theta}{\partial x^n} + N_n^k \frac{\partial \theta}{\partial y^n} = -C_1 p_n. \tag{B.91}$$

Indeed,

$$N_n^k \frac{\partial \theta}{\partial y^k} = \frac{1}{K} N_n^k m_k = -\partial_n^* \theta + \frac{1}{K^2} T N_n^k (b n_k - n b_k) - a^k{}_{nj} y^j \frac{\partial \theta}{\partial y^k},$$

or

$$N_n^k \frac{\partial \theta}{\partial y^k} = -\partial_n^* \theta + \frac{1}{K^2} T (S^2 - U c^2 B) p_n - a^k{}_{nj} y^j \frac{\partial \theta}{\partial y^k},$$

which can also be written as follows:

$$N_n^k \frac{\partial \theta}{\partial y^k} = -\partial_n^* \theta + \frac{1}{K^2} T S^2 p_n - C_1 p_n - a^k{}_{nj} y^j \frac{\partial \theta}{\partial y^k}. \tag{B.92}$$

On the other hand,

$$\frac{\partial \theta}{\partial x^n} = \partial_n^* \theta - \frac{1}{K^2} T S^2 p_n + a^k{}_{nj} y^j \frac{\partial \theta}{\partial y^k}, \tag{B.93}$$

where (B.83) has been used. By adding this equality to (B.92) we just conclude that the choice (B.85) made for the coefficients N_n^k entails the angle-preserving property $d_n\theta = k_n$ (indicated in (2.8)) with the choice $k_n = -C_1 p_n$.

Let us insert (B.88) and (B.93) in (B.85), which yields

$$N_n^k = \left[\left(\frac{1}{c^2} b - U c^2 B_1\right) n^k + n\left(U - \frac{1}{c^2}\right) b^k\right] p_n + l^k \left[g n q p_n \frac{K}{B} + a^h{}_{nj} y^j l_h\right]$$

$$-K m^k \left[\frac{1}{K^2} T S^2 p_n - a^h{}_{nj} l^j m_h\right] - l^k \partial_n K - K m^k \partial_n \theta - a^k{}_{nj} y^j.$$

Noting the identity $l^k l_h + m^k m_h = \delta_h^k$ and the representation (B.90) of the function U, we get

$$N_n^k = \frac{1}{c^2}(b n^k - n b^k) p_n + l^k g n q p_n \frac{K}{B} - K m^k \frac{1}{K^2} T S^2 p_n - l^k \partial_n K - K m^k (\partial_n \theta + C_1 p_n).$$

Using here the formula (B.40) which describes the structure of the vector m^k, we obtain

$$N_n^k = \left[B \frac{1}{c^2}(b n^k - n b^k) + y^k g n q\right] \frac{1}{B} p_n - \left(\frac{1}{c^2}(b n^k - n b^k) + g q n^k\right) \frac{1}{B} S^2 p_n$$

$$-l^k\partial_n K - Km^k(\partial_n\theta + C_1 p_n)$$

$$= \left[gbq\frac{1}{c^2}(bn^k - nb^k) + y^k gnq\right]\frac{1}{B}p_n - gqn^k\frac{1}{B}S^2p_n - l^k\partial_n K - Km^k(\partial_n\theta + C_1p_n)$$

$$= \left[gbq\frac{1}{c^2}bn^k + nn^k gnq\right]\frac{1}{B}p_n - gqn^k\frac{1}{B}S^2p_n - l^k\partial_n K - Km^k(\partial_n\theta + C_1p_n),$$

so that the final representation is merely

$$N_n^k = -l^k\frac{\partial K}{\partial x^n} - Km^kP_n \tag{B.94}$$

with

$$P_n = \frac{\partial\theta}{\partial x^n} + C_1p_n. \tag{B.95}$$

We have arrived at the coefficients N_n^k which are tantamount to the coefficients announced in (2.6).

With the coefficients N_n^k indicated in (B.85), we obtain the values

$$d_ib = nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta \tag{B.96}$$

and

$$d_in = -Uc^2B_1p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}B_1\partial_i^*\theta. \tag{B.97}$$

Also, using the variable

$$q = \sqrt{\frac{1-c^2}{c^2}b^2 + n^2},$$

we get

$$qd_iq = \frac{1-c^2}{c^2}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right] + n\left[-Uc^2B_1p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}B_1\partial_i^*\theta\right].$$

For the square

$$S^2 = b^2 + q^2$$

the equality

$$\frac{1}{2}d_iS^2 = \frac{1}{c^2}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right] + n\left[-Uc^2B_1p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}B_1\partial_i^*\theta\right]$$

is obtained, so that

$$\frac{1}{2}d_iS^2 = -Uc^2gnqp_i - S^2\frac{1}{K}\partial_i^*K - c\sqrt{\frac{q}{\nu}}gnq\partial_i^*\theta = -S^2\frac{1}{K}\partial_i^*K - c\sqrt{\frac{q}{\nu}}gnqP_i \tag{B.98}$$

and

$$qd_iq = -q^2\frac{1}{K}\partial_i^*K - c\sqrt{\frac{q}{\nu}}(b+gq)nP_i. \tag{B.99}$$

It can readily be verified that

$$d_i\frac{n}{q} = -\frac{1-c^2}{c^2q^3}bBc\sqrt{\frac{q}{\nu}}P_i.$$

We directly obtain the equalities

$$d_i(bq) = \frac{b}{q}\frac{1}{c^2}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right] - \frac{b}{q}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$+\frac{b}{q}n\left[-Uc^2B_1p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}B_1\partial_i^*\theta\right] + qnUc^2p_i - q\frac{1}{K}b\partial_i^*K + qc\sqrt{\frac{q}{\nu}}n\partial_i^*\theta,$$

or

$$d_i(bq) = -\frac{b}{q}b\left[nUc^2p_i + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right] + \frac{b}{q}n\left[-Uc^2gqp_i - c\sqrt{\frac{q}{\nu}}gq\partial_i^*\theta\right]$$

$$+qnUc^2p_i - 2bq\frac{1}{K}\partial_i^*K + qc\sqrt{\frac{q}{\nu}}n\partial_i^*\theta,$$

and

$$d_iB = -g\frac{1}{q}nBUc^2p_i - g\frac{1}{q}Bc\sqrt{\frac{q}{\nu}}n\partial_i^*\theta - 2B\frac{1}{K}\partial_i^*K + bqg_i, \tag{B.100}$$

together with

$$d_i\frac{b}{q} = -\frac{b}{q^3}\frac{1}{c^2}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right] + \frac{b}{q^3}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$-\frac{b}{q^3}n\left[-Uc^2B_1p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}B_1\partial_i^*\theta\right] + \frac{1}{q}nUc^2p_i - \frac{1}{q}\frac{1}{K}b\partial_i^*K + \frac{1}{q}c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta,$$

which is

$$d_i\frac{b}{q} = \frac{b}{q^3}\frac{1}{c^2}b\frac{1}{K}b\partial_i^*K + \frac{b}{q^3}b\left[-\frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$+\frac{b}{q^3}n\left[\frac{1}{K}n\partial_i^*K + c\sqrt{\frac{q}{\nu}}gq\partial_i^*\theta\right] + \frac{1}{q^3}nBUc^2p_i - \frac{1}{q}\frac{1}{K}b\partial_i^*K + \frac{1}{q}c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta,$$

coming to

$$d_i\frac{b}{q} = \frac{n}{q^3}B\left(Uc^2p_i + c\sqrt{\frac{q}{\nu}}\partial_i^*\theta\right) = \frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}\left(C_1p_i + \partial_i^*\theta\right). \tag{B.101}$$

Applying the operator d_i to the scalar T introduced by (B.38) yields

$$d_iT = T\left(\frac{1}{2}\frac{q}{\nu}d_i\frac{\nu}{q} - \frac{1}{B}d_iB\right) = T\frac{1}{2}\frac{q}{\nu}(1-c^2)\left(g_i\frac{b}{q} + g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}\left(C_1p_i + \partial_i^*\theta\right)\right)$$

$$-T\frac{1}{B}\left(-g\frac{1}{q}nBUc^2p_i - g\frac{1}{q}Bc\sqrt{\frac{q}{\nu}}n\partial_i^*\theta - 2B\frac{1}{K}\partial_i^*K + bqg_i\right),$$

or

$$\frac{1}{T}d_iT = \frac{bq}{2B}\left(\frac{1}{X} - 4\right)g_i + 2\frac{1}{K}\partial_i^*K + g\frac{n}{2q}\frac{1}{X}\left(Uc^2p_i + c\sqrt{\frac{q}{\nu}}\partial_i^*\theta\right). \tag{B.102}$$

We also obtain

$$d_h\left[\left(\frac{n}{q}\right)^2\frac{q}{\nu}\right] = -2\frac{n}{q}\frac{q}{\nu}\frac{1-c^2}{c^2q^3}bBc\sqrt{\frac{q}{\nu}}P_h - \left(\frac{n}{q}\right)^2\left(\frac{q}{\nu}\right)^2d_h\frac{\nu}{q},$$

or

$$d_h\left[\left(\frac{n}{q}\right)^2\frac{q}{\nu}\right] = -2\frac{n}{\nu}\frac{1-c^2}{c^2q^3}bBc\sqrt{\frac{q}{\nu}}P_h - \left(\frac{n}{\nu}\right)^2(1-c^2)\left(g_h\frac{b}{q} + g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}P_h\right),$$

and

$$d_h\frac{1}{X} = 2\frac{n}{\nu}\frac{1-c^2}{q^3}bBc\sqrt{\frac{q}{\nu}}P_h + \left(\frac{n}{\nu}\right)^2c^2(1-c^2)\left(g_h\frac{b}{q} + g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}P_h\right).$$

Next, starting from (B.40), it follows that

$$d_i(Km^n) = -Km^n\frac{1}{2}\frac{q}{\nu}d_i\frac{\nu}{q} + c\sqrt{\frac{q}{\nu}}d_i\left(\frac{1}{c^2}(bn^n - nb^n) + gqn^n\right),$$

or

$$d_i(Km^n) = -Km^n\frac{1}{2}\frac{q}{\nu}(1-c^2)\left(g_i\frac{b}{q} + g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}\left(C_1p_i + \partial_i^*\theta\right)\right)$$

$$+c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}n^n\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$-c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}b^n\left[-Uc^2\left(\frac{1}{c^2}b + gq\right)p_i - \frac{1}{K}n\partial_i^*K - c\sqrt{\frac{q}{\nu}}\left(\frac{1}{c^2}b + gq\right)\partial_i^*\theta\right]$$

$$+c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}(b\nabla_in^n - n\nabla_ib^n) + c\sqrt{\frac{q}{\nu}}\left(gq\nabla_in^n + g_iqn^n\right)$$

$$+c\sqrt{\frac{q}{\nu}}gn^n\frac{1}{q}\frac{1-c^2}{c^2}b\left[nUc^2p_i - \frac{1}{K}b\partial_i^*K + c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$+c\sqrt{\frac{q}{\nu}}gn^n\frac{1}{q}n\left[-Uc^2\left(\frac{1}{c^2}b+gq\right)p_i-\frac{1}{K}n\partial_i^*K-c\sqrt{\frac{q}{\nu}}\left(\frac{1}{c^2}b+gq\right)\partial_i^*\theta\right],$$

which is transformed to

$$d_i(Km^n)=-Km^n\frac{1}{2}\frac{q}{\nu}(1-c^2)\left(g_i\frac{b}{q}+g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}\left(C_1p_i+\partial_i^*\theta\right)\right)$$

$$+c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}n^n\left[-\frac{1}{K}b\partial_i^*K+c\sqrt{\frac{q}{\nu}}n\partial_i^*\theta\right]$$

$$-c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}b^n\left[-\frac{1}{K}n\partial_i^*K-c\sqrt{\frac{q}{\nu}}\left(\frac{1}{c^2}b+gq\right)\partial_i^*\theta\right]$$

$$+c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}(b\nabla_i n^n-n\nabla_i b^n)+c\sqrt{\frac{q}{\nu}}\left(gq\nabla_i n^n+g_iqn^n\right)+c\frac{q}{\nu}gn^n\frac{1-c^2}{qc^2}bcn\partial_i^*\theta$$

$$+c\sqrt{\frac{q}{\nu}}gn^n\frac{1}{q}\left[-\frac{1}{K}q^2\partial_i^*K-nc\sqrt{\frac{q}{\nu}}\left(\frac{1}{c^2}b+gq\right)\partial_i^*\theta\right]$$

$$+Uc^2c\sqrt{\frac{q}{\nu}}\frac{1}{c^2}\left[\frac{n}{q}(q-gc^2(b+gq))n^n+b^n\left(\frac{1}{c^2}b+gq\right)\right]p_i.$$

Notice that

$$b\nabla_i n^n-n\nabla_i b^n=-y^np_i,\qquad \nabla_i n^n=-\frac{1}{c^2}b^np_i.$$

In this way we straightforwardly obtain the representation

$$d_i(Km^n)=-Km^n\frac{1}{2}\frac{q}{\nu}(1-c^2)\left(g_i\frac{b}{q}+g\frac{n}{q^3}Bc\sqrt{\frac{q}{\nu}}\,P_i\right)-\frac{1}{c}\sqrt{\frac{q}{\nu}}\,y^np_i+c\sqrt{\frac{q}{\nu}}g_iqn^n$$

$$-c\sqrt{\frac{q}{\nu}}\,\frac{1}{c^2}gqb^np_i-m^n\partial_i^*K+\frac{q}{\nu}\left[y^n-\frac{n}{q}gc^2(b+gq)n^n+gqb^n\right]P_i. \tag{B.103}$$

We can write the coefficients (B.85) in terms of the vector m^k

$$N_n^k=-Km^k(C_1p_n+\partial_n^*\theta)+\frac{1}{c^2}(bn^k-nb^k)p_n-l^k\partial_n^*K-a^k{}_{nj}y^j \tag{B.104}$$

and after that represent them in the form

$$N_n^k=-\left(N_{\{l\}}l^k+N_{\{m\}}m^k\right)Kp_n-l^k\partial_n^*K-Km^k\partial_n^*\theta-a^k{}_{nj}y^j \tag{B.105}$$

with

$$N_{\{l\}} = g\frac{nq}{B}, \qquad N_{\{m\}} = C_1 - \frac{1}{c}\sqrt{\frac{\nu}{q}}\frac{S^2}{B}. \tag{B.106}$$

Differentiating the coefficients N^k_n given by (B.85) leads to the representation

$$N^k_{nm} = \left[b_m\left(\frac{1}{c^2} - U\frac{\nu}{q}\right)n^k + n_m\left(U - \frac{1}{c^2}\right)b^k\right]p_n - gUc^2\frac{n}{q}n_m n^k p_n$$

$$+\frac{n}{2q^2}\frac{U}{\nu}g(1-c^2)z^k(bn_m - nb_m)p_n - a^k{}_{nm}$$

$$-l^k\partial^*_n l_m + \left(-l_m m^k + l^k m_m - gcm^k m_m\frac{1}{2X}\frac{n}{\sqrt{q\nu}}\right)\partial^*_n\theta - m^k\partial^*_n m_m.$$

Here,

$$z^k = nb^k - (b+gc^2q)n^k = -\sqrt{\frac{\nu}{q}}cKm^k, \qquad U = C_1\frac{1}{c}\sqrt{\frac{q}{\nu}},$$

so that we can write the above coefficients as follows:

$$N^k_{nm} = \left[b_m\left(\frac{1}{c^2} - C_1\frac{1}{c}\sqrt{\frac{\nu}{q}}\right)n^k + n_m\left(C_1\frac{1}{c}\sqrt{\frac{q}{\nu}} - \frac{1}{c^2}\right)b^k\right]p_n - gC_1\frac{1}{c}\sqrt{\frac{q}{\nu}}c^2\frac{n}{q}n_m n^k p_n$$

$$-\frac{n}{2q^2}\frac{1}{\nu}g(1-c^2)C_1Km^k(bn_m - nb_m)p_n - a^k{}_{nm}$$

$$-l^k\partial^*_n l_m + \left(-l_m m^k + l^k m_m - gcm^k m_m\frac{1}{2X}\frac{n}{\sqrt{q\nu}}\right)\partial^*_n\theta - m^k\partial^*_n m_m, \tag{B.107}$$

or

$$N^k{}_{nm} = \left(-l_m m^k + l^k m_m - gcm^k m_m\frac{1}{2X}\frac{n}{\sqrt{q\nu}}\right)P_n - l^k\partial^*_n l_m - m^k\partial^*_n m_m$$

$$+\frac{1}{c^2}(b_m n^k - n_m b^k)p_n - a^k{}_{nm}. \tag{B.108}$$

This representation can be written in the form similar to (2.11).

From the previous representation the coefficients $N^k_{nmi} = \partial N^k_{nm}/\partial y^i$ are evaluated to read

$$N^k_{nmi} = \frac{1}{K}Z_n m^k m_m m_i \tag{B.109}$$

with

$$Z_n = -\frac{3}{4}g(1-c^2)^2B^2\frac{1}{q^3\nu^3}(2b\nu + gc^2n^2)P_n + \frac{1}{2}nc\partial^*_n\left(g\frac{1}{X}\frac{1}{\sqrt{q\nu}}\right). \tag{B.110}$$

The last two formulas agree completely with the general formula (2.14) indicated in Section 2.

The following equalities

$$\frac{\partial\dfrac{\nu}{q}}{\partial y^i} = (1-c^2)g\frac{n}{q^3}(nb_i - bn_i), \qquad \frac{\partial\dfrac{n}{\nu}}{\partial y^i} = \frac{1}{q\nu^2}\frac{1}{c^2}(1-c^2)(b+gc^2q)(bn_i - nb_i),$$

$$\frac{\partial\dfrac{1}{X}}{\partial y^i} = -(1-c^2)(2b\nu + gc^2n^2)\frac{1}{\nu^2}\frac{n}{q^3}cB\frac{1}{K}\sqrt{\frac{q}{\nu}}m_i,$$

$$\frac{\partial\left(\dfrac{n^2}{q\nu}\right)}{\partial y^i} = \frac{1}{c^2}(1-c^2)\frac{n}{\nu}(bn_i - nb_i)\frac{1}{q^3\nu}(2b\nu + gc^2n^2),$$

and

$$\frac{\partial A^hA_h}{\partial y^i} = \frac{3g^2}{4X}(1-c^2)^2(2b\nu + gc^2n^2)\frac{1}{\nu^3}\frac{1}{q^3}B^2\frac{1}{K}\frac{cn}{\sqrt{q\nu}}m_i$$

are convenient to use in process of derivation of the Z_n indicated above.

It is easy again to perform the evaluation of the curvature tensor making the choice of the vector m^k in accordance with (B.38)-(B.40). With arbitrary smooth $g = g(x)$ and $c = c(x)$, the result reads

$$M^n{}_{ij} = d_iN^n_j - d_jN^n_i = Km^nM_{ij}, \qquad M_{ij} = \frac{\partial k_j}{\partial x^i} - \frac{\partial k_i}{\partial x^j} \tag{B.111}$$

and

$$E_k{}^n{}_{ij} = -\left[l_km^n - l^nm_k + gcm^nm_k\frac{1}{2X}\frac{n}{\sqrt{q\nu}}\right]M_{ij} \equiv -\frac{\partial M^n{}_{ij}}{\partial y^k} \tag{B.112}$$

((B.55) has been applied), so that the curvature tensor

$$\rho_k{}^n{}_{ij} = E_k{}^n{}_{ij} - M^h{}_{ij}C^n{}_{hk} \tag{B.113}$$

is simply

$$\rho_k{}^n{}_{ij} = (l^nm_k - l_km^n)M_{ij}. \tag{B.114}$$

Let us find

$$l_nm_k - l_km_n = T\frac{1}{B}\left[(u_n + gqb_n)(bn_k - nb_k) - (u_k + gqb_k)(bn_n - nb_n)\right]$$

$$= T\frac{1}{B}\left[\widetilde{b}^2(b_nn_k - b_kn_n) - n^2(b_kn_n - b_nn_k) + gqb(b_nn_k - b_kn_n)\right].$$

Thus,

$$l_nm_k - l_km_n = T(b_nn_k - b_kn_n) \tag{B.115}$$

and

$$\rho_{knij} = T(b_nn_k - b_kn_n)M_{ij}, \tag{B.116}$$

so that the factor function f_1 indicated in (4.20) is equal to cT.

Appendix C: Randers metric in two-dimensional case

The Finsler metric function F is now of the form

$$F = S + b. \tag{C.1}$$

All the formulas (B.1)–(B.11) can be applied. We obtain $\det(g_{ij}) = (F/S)^3 \det(a_{ij})$, so that the formulas (B.38)–(B.40) are to be replaced by

$$T = \frac{1}{c}\sqrt{\frac{\det(g_{ij})}{\det(a_{mn})}} = \frac{1}{c}\sqrt{\left(\frac{F}{S}\right)^3} \tag{C.2}$$

and

$$m_i = \frac{1}{F}T(bn_i - nb_i) = \frac{1}{cS}\sqrt{\frac{F}{S}}\,(bn_i - nb_i), \tag{C.3}$$

together with $m^i = g^{ij}m_j$ and

$$m^i = \frac{1}{cF}\sqrt{\frac{S}{F}}\left(-nb^i + (b + c^2S)n^i\right). \tag{C.4}$$

The partial derivative $\partial F/\partial x^n$ of the F given by (C.1) is obviously regular of class C^∞ with respect to the variable y.

Again, we can use the atlas (B.62)–(B.63) and the formulas of the type (B.64)–(B.70). In place of (B.77) we obtain the representation

$$\widetilde{\Theta} = \int \frac{b^2}{F^2} T d\widetilde{w} = \frac{1}{c}\int \frac{1}{\sqrt{1+\sqrt{1+w^2}}\left(\sqrt{1+w^2}\right)^3} d\widetilde{w},$$

or

$$\widetilde{\Theta} = \frac{1}{c}\int \frac{1}{\sqrt{1+\sqrt{\frac{1}{c^2}+\widetilde{w}^2}}\left(\sqrt{\frac{1}{c^2}+\widetilde{w}^2}\right)^3} d\widetilde{w}, \tag{C.5}$$

where the integration constant is assumed to be subjected to the condition $\widetilde{\Theta}_{|\widetilde{w}=0} = 0$ to agree with $\theta(x, b(x)) = 0$.

It follows that

$$\frac{\partial\theta}{\partial x^n} = \frac{\partial\widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} + \frac{\partial\widetilde{\Theta}}{\partial\widetilde{w}}\frac{\partial\widetilde{w}}{\partial x^n} = \frac{\partial\widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} + \frac{1}{F^2}Tb^2\left(\frac{\partial\widetilde{w}}{\partial x^n} - a^k{}_{nj}y^j(bn_k - nb_k)\frac{1}{b^2}\right) + a^k{}_{nj}y^j\frac{\partial\theta}{\partial y^k}, \tag{C.6}$$

so that

$$\frac{\partial\theta}{\partial x^n} = \frac{\partial\widetilde{\Theta}}{\partial c}\frac{\partial c}{\partial x^n} - \frac{1}{F^2}T\left(S^2cn^h\nabla_n\widetilde{b}_h + bn\frac{1}{c}\frac{\partial c}{\partial x^n}\right) + a^k{}_{nj}y^j\frac{\partial\theta}{\partial y^k}. \tag{C.7}$$

Here, all the terms are smooth of class C^∞ regarding the y-dependence. The same conclusion can be arrived at in the regions $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$.

The factor-relation (4.19) for the curvature tensor takes now on the form

$$\rho_{knij} = \sqrt{\left(\frac{F}{S}\right)^3}\,\bar{L}_{knij} \quad \text{with} \quad \bar{L}_{knij} = a_{nh}\bar{L}_k{}^h{}_{ij} \equiv \bar{L}_{knij}(x). \tag{C.8}$$

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